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1 Final Good and Processed Good Sectors

1.1 Final Good Sector

The final good sector is perfectly competitive with Cobb-Douglas production:

$$Y = X_1^{\epsilon} X_2^{1-\epsilon} \tag{1}$$

Final good producers maximize profit by choosing the amount of processed good given the prices of those goods. Let Y be the numeraire. The Lagrangean for final good producers is

$$\pi_Y = Y - P_{X_1} X_1 - P_{X_2} X_2 \tag{2}$$

The first order conditions are

$$\frac{\partial \pi_Y}{\partial X_1} = \epsilon X_1^{\epsilon-1} X_2^{1-\epsilon} - P_{X_1} = 0 \tag{3}$$

$$\frac{\partial \pi_Y}{\partial X_2} = (1-\epsilon) X_1^{\epsilon} X_2^{-\epsilon} - P_{X_2} = 0 \tag{4}$$

from which we obtain the demand functions

$$X_1 = (\frac{\epsilon}{P_{X_1}})^{\frac{1}{1-\epsilon}} X_2 \tag{5}$$

$$X_2 = (\frac{1-\epsilon}{P_{X_2}})^{\frac{1}{\epsilon}} X_1 \tag{6}$$

The inverse demand functions for $X_i's$ are

$$P_{X_1} = \epsilon X_1^{\epsilon-1} X_2^{1-\epsilon} \tag{7}$$

$$P_{X_2} = (1-\epsilon) X_1^{\epsilon} X_2^{-\epsilon} \tag{8}$$

which together yield

$$\frac{P_{X_1}}{P_{X_2}} = \frac{\epsilon}{1-\epsilon} (\frac{X_1}{X_2})^{-1} \tag{9}$$

The competitive final-good producer pays compensation ϵY and $(1-\epsilon)Y$ to the processed-good 1 and processed-good 2. So we get $\epsilon Y = P_{X_1} X_1$ and $(1-\epsilon)Y = P_{X_2} X_2$.

1.2 Processed Good Sector

The processed good sector also is perfectly competitive. The production functions for the two processed good industries X_1 and X_2 are

$$\begin{aligned} X_1 &= \int_0^{N_1} G_{1j}^{\lambda} \left(Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j} \right)^{1-\lambda} dj, \quad 0 < \lambda, \delta < 1 \\ X_2 &= \int_0^{N_2} G_{2j}^{\lambda} \left(Z_{2j}^{\delta} Z_2^{\gamma} Z_1^{1-(\delta+\gamma)} l_{2j} \right)^{1-\lambda} dj, \quad 0 < \lambda, \delta < 1 \end{aligned} \tag{10}$$

where $Z_1 = (1/N_1) \int_0^{N_1} Z_{1j} dj$, and $Z_2 = (1/N_1) \int_0^{N_2} Z_{2j} dj$ are the average qualities of the intermediate goods G_{1j} and G_{2j}. We assume labor moves freely across firms and industries, so wages must be equal everywhere. Taking the prices of intermediate goods and the wage as given, processed goods producers in the first (i.e., X_1) industry choose the amounts of the G_{1j} and labor l_{1j} to maximize profit:

$$\pi_{X_1} = P_{X_1} X_1 - \int_0^{N_1} P_{G_{1j}} G_{1j} dj - \int_0^{N_1} w l_{1j} dj$$

The first order conditions are

$$\begin{aligned}
\frac{\partial \pi_{X_1}}{\partial G_{1j}} &= P_{X_1}(\frac{\partial X_1}{\partial G_{1j}}) - P_{G_{1j}} \\
&= P_{X_1}\lambda G_{1j}^{\lambda-1}(Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j})^{1-\lambda} - P_{G_{1j}} \\
&= 0 \\
\frac{\partial \pi_{X_1}}{\partial l_{1j}} &= P_{X_1}(\frac{\partial X_1}{\partial l_{1j}}) - w_{1j} \\
&= P_{X_1}G_{1j}^{\lambda}(Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j})^{-\lambda}(1-\lambda)Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j} - w_{1j} \\
&= 0
\end{aligned}$$

from which we obtain

$$G_{1j} = [\frac{\lambda P_{X_1}}{P_{G_{1j}}}]^{\frac{1}{1-\lambda}} Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j} \tag{11}$$

$$l_{1j} = (P_{X_1}\frac{1-\lambda}{w})^{\frac{1}{\lambda}}G_{1j}(Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)})^{\frac{1-\lambda}{\lambda}} \tag{12}$$

Carrying out the same steps for the X_2 industry gives the profit expression

$$\pi_{X_2} = P_{X_2}X_2 - \int_0^{N_2} P_{G_{2j}}G_{2j}dj - \int_0^{N_2} wl_{2j}dj$$

and the solutions

$$G_{2j} = [\frac{\lambda P_{X_2}}{P_{G_{2j}}}]^{\frac{1}{1-\lambda}} Z_{2j}^{\delta}Z_2^{\gamma}Z_1^{1-(\delta+\gamma)}l_{1j} \tag{13}$$

$$l_{2j} = (P_{X_2}\frac{1-\lambda}{w})^{\frac{1}{\lambda}}G_{2j}(Z_{2j}^{\delta}Z_2^{\gamma}Z_1^{1-(\delta+\gamma)})^{\frac{1-\lambda}{\lambda}} \tag{14}$$

Substituting the inverse demands for X_1 and X_2, given by 7 and 8, into 11 and 13 gives the demand functions for the G_{ij}:

$$\begin{aligned}
G_{1j} &= [\frac{\lambda\epsilon(\frac{X_1}{X_2})^{\epsilon-1}}{P_{G_{1j}}}]^{\frac{1}{1-\lambda}} Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j} \\
G_{2j} &= [\frac{\lambda(1-\epsilon)(\frac{X_1}{X_2})^{\epsilon}}{P_{G_{2j}}}]^{\frac{1}{1-\lambda}} Z_{2j}^{\delta}Z_2^{\gamma}Z_1^{1-(\delta+\gamma)}l_{2j}
\end{aligned}$$

We next derive the allocation of labor between the two processed good industries. Profits of processed goods firms are zero, so

$$\begin{aligned}
P_{X_1}X_1 &= \int_0^{N_1}(P_{G_{1j}}G_{1j} + wl_{1j})dj \\
P_{X_2}X_2 &= \int_0^{N_2}(P_{G_{2j}}G_{2j} + wl_{2j})dj
\end{aligned} \tag{15}$$

Competition and Cobb-Douglas production in the processed good sector imply that the processed good producers pay $\lambda P_{X_i}X_i$, $i = 1, 2$ to the intermediate good producers, so $\lambda P_{X_1}X_1 = \int_0^{N_1} G_{1j}P_{G_{1j}}dj$ and $\lambda P_{X_2}X_2 = \int_0^{N_2} G_{2j}P_{G_{2j}}dj$. They pay $(1-\lambda)P_{X_i}X_i$ to the workers, so $(1-\lambda)P_{X_1}X_1 = \int_0^{N_1} wl_{1j}dj$ and

$(1-\lambda)P_{X_2}X_2 = \int_0^{N_2} wl_{2j}dj$. Similarly, competition and Cobb-Douglas production in the final good sector imply that $P_{X_1}X_1 = \epsilon Y$ and $P_{X_2}X_2 = (1-\epsilon)\,Y$. We therefore have

$$\begin{aligned}
\lambda\epsilon Y &= \int_0^{N_1} G_{1j}P_{G_{1j}}dj \\
\lambda(1-\epsilon)Y &= \int_0^{N_2} G_{2j}P_{G_{2j}}dj \\
(1-\lambda)\epsilon Y &= \int_0^{N_1} wl_{1j}dj \\
(1-\lambda)(1-\epsilon)Y &= \int_0^{N_2} wl_{2j}dj
\end{aligned}$$

Define the total amount of labor in industries X_1 and X_2 as $L_1 \equiv \int_0^{N_1} l_{1j}dj$ and $L_2 \equiv \int_0^{N_2} l_{2j}dj$. Then we get the labor allocation from the ratio of the total labor compensation paid in the two industries:

$$\frac{(1-\lambda)\epsilon Y}{(1-\lambda)(1-\epsilon)Y} = \frac{\int_0^{N_1} wl_{1j}dj}{\int_0^{N_2} wl_{2j}dj} = \frac{w\int_0^{N_1} l_{1j}dj}{w\int_0^{N_2} l_{2j}dj} = \frac{\int_0^{N_1} l_{1j}dj}{\int_0^{N_2} l_{2j}dj} = \frac{L_1}{L_2} = \frac{\epsilon}{1-\epsilon} \tag{16}$$

We can also get the above result by setting $MPl_{1j} = MPl_{2j} \equiv w$. That derivation is omitted here.

1.3 Derivation of $\frac{X_1}{X_2}$

The ratio X_1/X_2 appears in 9 and elsewhere, so we obtain an expression for it.

$$\begin{aligned}
X_1 &= \int_0^{N_1} G_{1j}^{\lambda}\left(Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j}\right)^{1-\lambda}dj \\
&= \int_0^{N_1} [\frac{\lambda\epsilon(\frac{X_1}{X_2})^{\epsilon-1}}{P_{G_{1j}}}]^{\frac{\lambda}{1-\lambda}}\,(Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j})^{\lambda}(Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j})^{1-\lambda}dj \qquad (17) \\
&= \int_0^{N_1} [\frac{\lambda\epsilon(\frac{X_1}{X_2})^{\epsilon-1}}{P_{G_{1j}}}]^{\frac{\lambda}{1-\lambda}}\,Z_{1j}^{\delta}Z_1^{\gamma}Z_2^{1-(\delta+\gamma)}l_{1j}dj
\end{aligned}$$

Similarly, we get

$$X_2 = \int_0^{N_2} [\frac{\lambda(1-\epsilon)(\frac{X_1}{X_2})^{\epsilon}}{P_{G_{2j}}}]^{\frac{\lambda}{1-\lambda}}\,(Z_{2j}^{\delta}Z_2^{\gamma}Z_1^{1-(\delta+\gamma)}l_{2j})dj \tag{18}$$

Taking the ratio of 17 to 18, we get

$$\begin{aligned}
\frac{X_1}{X_2} &= (\frac{\epsilon}{1-\epsilon})^{\frac{\lambda}{1-\lambda}}(\frac{X_1}{X_2})^{-\frac{\lambda}{1-\lambda}}(\frac{Z_1}{Z_2})^{(\delta+\gamma)-1+\gamma}\frac{\int_0^{N_1} Z_{1j}^{\delta}l_{1j}P_{G_{1j}}^{\frac{-\lambda}{1-\lambda}}dj}{\int_0^{N_2} Z_{2j}^{\delta}l_{2j}P_{G_{2j}}^{-\frac{\lambda}{1-\lambda}}dj} \\
\Rightarrow\ (\frac{X_1}{X_2})^{\frac{1}{1-\lambda}} &= (\frac{\epsilon}{1-\epsilon})^{\frac{\lambda}{1-\lambda}}(\frac{Z_1}{Z_2})^{\delta+2\gamma-1}\frac{\int_0^{N_1} l_{1j}P_{G_{1j}}^{\frac{-\lambda}{1-\lambda}}Z_{1j}^{\delta}dj}{\int_0^{N_2} l_{2j}P_{G_{2j}}^{-\frac{\lambda}{1-\lambda}}Z_{2j}^{\delta}dj} \\
\Rightarrow \frac{X_1}{X2} &= (\frac{\epsilon}{1-\epsilon})^{\lambda}(\frac{Z_1}{Z_2})^{(\delta+2\gamma-1)(1-\lambda)}(\frac{\int_0^{N_1} l_{1j}P_{G_{1j}}^{-\frac{\lambda}{1-\lambda}}Z_{1j}^{\delta}dj}{\int_0^{N_2} l_{2j}P_{G_{2j}}^{-\frac{\lambda}{1-\lambda}}Z_{2j}^{\delta}dj})^{1-\lambda} \qquad (19)
\end{aligned}$$

2 Intermediate Goods Sector

The intermediate goods sector is monopolistically competitive, and each firm has the power to set its own price. The two industries in the sector are similar in structure, but notational complexities make it expedient to discuss the industries separately.

2.1 Incumbents in Industry 1

The fixed operating cost is the same for all firms in the industry, $\theta_1 Z_1^2/Z_2$, where Z_1 and Z_2 are the average quality levels in the two industries and are taken as given by individual firms. The unit cost for industry 1 is A_1, i.e. 1 unit of intermediate good requires A_1 units of final good. Firm profit is

$$F_{1j} = G_{1j}\left(P_{G_{1j}} - A_1\right) - \theta_1 \frac{Z_1^2}{Z_2} \tag{20}$$

Given the demand function, each firm sets its own price and devotes R&D resource to maximize the present value of profit. The current value of Hamiltonian is

$$CVH_{1j} = G_{1j}(P_{G_{1j}} - A_1) - \theta_1 \frac{Z_1^2}{Z_2} - R_{1j} + q_{1j}(\alpha_1 I_{1j}) \tag{21}$$

s.t. $G_{1j} = [\lambda\epsilon(\frac{X_1}{X_2})^{\epsilon-1}/P_{G_{1j}}]^{\frac{1}{1-\lambda}} Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j}$ and $\dot{Z}_{1j} = \alpha_1 I_{1j}$.

Define r_1 as the return to R&D in industry 1 and r_2 as the return in R&D in industry 2. Investment I_1 is determined as follows:

$$\begin{cases} I_{1j} = 0 & \text{if } r_1 < r_2 \\ I_{1j} = R_{1j} & \text{if } r_1 = r_2 \\ I_{2j} = R_{1j} + \frac{N_2}{N_1} R_{2j} & \text{if } r_1 > r_2 \end{cases}$$

Later we will see that the numbers of firms in each industry N_1 and N_2 always jump to make $r_1 = r_2$ and thus $I_{1j} = R_{1j}$.

The necessary conditions are:

1. $\dot{Z}_{1j} = \alpha_1 I_{1j}$
2. $\partial CVH_{1j}/\partial P_{G1j} = 0$
3. $\partial CVH_{1j}/\partial R_{1j} = -1 + q_{1j}\alpha_1$
4. $\dot{q_{1j}} = r_1 q_{1j} - \partial CVH_{1j}/\partial Z_{1j}$
5. $Z_{1j,t=0}$ is given
6. $lim_{t\to\infty} e^{-\int_t^\tau r(s)ds} q_{1j}(t) Z_{1j}(t) = 0$

Condition (2) provides the solution for the price $P_{G_{1j}}$ of intermediate good G_{1j}. Recall that $G_{1j} = [\lambda P_{X_1}/P_{G_{1j}}]^{\frac{1}{1-\lambda}} Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j}$. Then

$$\begin{aligned} 0 &= \frac{\partial CVH_{1j}}{\partial P_{G1j}} \Rightarrow 0 = \frac{\partial (P_{G_{1j}} - A_1) G_{1j}}{\partial P_{G_{1j}}} && \text{because no other parts of } CVH_{1j} \text{ include } P_{G_{1j}} \\ &= \frac{\partial \left[\left(P_{G_{1j}}^{1-\frac{1}{1-\lambda}} - A_1 P_{G_{1j}}^{-\frac{1}{1-\lambda}} \right) (\lambda P_{X_1})^{\frac{1}{1-\lambda}} Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j} \right]}{\partial P_{G_{1j}}} && \text{by substituting for } G_{1j} \\ &= \frac{-\lambda}{1-\lambda} P_{G_{1j}}^{-\frac{1}{1-\lambda}} + \frac{1}{1-\lambda} A_1 P_{G_{1j}}^{-\frac{1}{1-\lambda}-1} \end{aligned}$$

$$\Rightarrow 0 = \frac{-\lambda}{1-\lambda} P_{G_{1j}}^{-\frac{1}{1-\lambda}} + \frac{1}{1-\lambda} A_1 P_{G_{1j}}^{-\frac{1}{1-\lambda}-1}$$

$$\begin{aligned} &\Rightarrow \quad P_{G_{1j}} = \frac{A_1}{\lambda} \\ &\equiv \quad P_{G_1} \end{aligned} \tag{22}$$

Each firm in industry 1 faces the same unit cost, A_1 units of Y, so the prices of all intermediate-good firms in industry 1 are the same monopoly markup over the same unit cost.

Condition (3) gives the rule for setting the value of R&D spending. Recall from above that we will show later that the rates of return to R&D in the two intermediate good industries, r_1 and r_2, always are equal consequently R&D investment I_{1j} by the firm producing G_{1j} always equals that firm's retained earnings R_{1j}. Then we obtain from condition (3)

$$\begin{aligned}
\frac{\partial CVH_{1j}}{\partial R_{1j}} &= -1 + q_{1j}\alpha_1 \\
&= 0 \\
&\Longrightarrow \begin{cases} R_{1j} = \infty & \text{if } 1/\alpha_1 > q_{1j} \\ R_{1j} > 0 & \text{if } 1/\alpha_1 = q_{1j} \\ R_{1j} = 0 & \text{if } 1/\alpha_1 < q_{1j} \end{cases}
\end{aligned}$$

The interior solution, $1/\alpha_1 = q_{1j}$, is the same for all j, implying that in the interior all firms in industry 1 choose the same level of R&D, which we denote R_1.

Rearranging condition (4) allows us to express the rate of return to R&D as the profit flow plus any capital gain:

$$\begin{aligned}
r_1 &= \frac{\partial CVH_{1j}/\partial Z_{1j}}{q_{1j}} + \frac{\dot{q_{1j}}}{q_{1j}} \\
&= \frac{\partial F_{1j}/\partial Z_{1j}}{q_{1j}} + \frac{\dot{q_{1j}}}{q_{1j}} \qquad (23)
\end{aligned}$$

Condition (6) is the transversality condition.

Substituting the demand function 11 into the cash flow 20 gives

$$\begin{aligned}
F_{1j} &= A_1(\frac{1}{\lambda} - 1)\{\frac{\lambda P_{X_1}}{P_{G_1}}\}^{\frac{1}{1-\lambda}}(Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j}) - \theta_1 \frac{Z_1^2}{Z_2} \\
&= A_1(\frac{1-\lambda}{\lambda})(\frac{\lambda P_{X_1}}{A_1/\lambda})^{\frac{1}{1-\lambda}}(Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j}) - \theta_1 \frac{Z_1^2}{Z_2} \qquad (24)
\end{aligned}$$

So

$$\frac{\partial F_{1j}}{\partial Z_{1j}} = \delta A_1 \frac{1-\lambda}{\lambda}(\frac{\lambda P_{X_1}}{A_1/\lambda})^{\frac{1}{1-\lambda}} Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j} \qquad (25)$$

Using 23 and 25, we can express the return in R&D in industry 1 as

$$\begin{aligned}
r_{1j} &= \frac{\partial F_{1j}}{\partial Z_{1j}} \frac{1}{q_{1j}} + \frac{\dot{q_{1j}}}{q_{1j}} \qquad (26) \\
&= \frac{\partial F_{1j}}{\partial Z_{1j}} \frac{1}{q_{1j}} + 0 \quad \text{because } q_{1j} = 1/\alpha_1, \text{ a constant} \\
&= \frac{\partial F_{1j}}{\partial Z_{1j}} \alpha_1 \\
&= \alpha_1 \delta A_1 \frac{1-\lambda}{\lambda}(\frac{\lambda P_{X_1}}{A_1/\lambda})^{\frac{1}{1-\lambda}} Z_{1j}^{\delta-1} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)} l_{1j}
\end{aligned}$$

We now prove symmetry among firms in the same industry. Plug the demand for G_{1j} 12 into the demand for labor l_{1j} 11 to obtain

$$w = (1-\lambda) P_{X1}(\frac{\lambda P_{X_1}}{P_{G1j}})^{\frac{\lambda}{1-\lambda}} Z_{1j}^{\delta} Z_1^{\gamma} Z_2^{1-(\delta+\gamma)}$$

All firms in industry 1 face the same spillovers, Z_1 and Z_2, and the same price of processed good, P_{X_1}. They also set the same price $P_{G_{1j}}$ according to 22. If $Z_{1j} > Z_{1k}$, then all workers would be allocated to G_{1j} and get paid a higher wage, leaving no demand for G_{1k}. The firm producing G_{1j} would take over the whole industry 1 as a monopoly, and its profit would exceed its fixed operating cost, generating an excess monopoly profit.[1] The entry cost is zero, however, so the monopoly profit immediately would induce new firms to enter industry 1 with new varieties carrying the average level of quality Z_1. The incumbent here is a monopoly, so the average quality level would be the monopoly's quality. Entry would continue to the point where profits had been driven down to fixed operating cost, resulting in a new monopolistically competitive equilibrium in which all firms would have the same quality and thus would divide industry 1 equally. In this way, the assumptions that entry is free and that new entrants have the average quality of existing varieties ensure the symmetry among firms in the industry.

Symmetry implies that $Z_{1j} = Z_1$ for all j, so 34 becomes

$$\begin{aligned}
\frac{X_1}{X_2} &= (\frac{\epsilon}{1-\epsilon})^{\lambda}(\frac{Z_1}{Z_2})^{(\delta+2\gamma-1)(1-\lambda)}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda}(\frac{\int_0^{N_1} Z_{1j}^{\delta} l_{1j} dj}{\int_0^{N_2} Z_{2j}^{\delta} l_{2j} dj})^{1-\lambda} \\
&= (\frac{\epsilon}{1-\epsilon})^{\lambda}(\frac{Z_1}{Z_2})^{(2\delta+2\gamma-1)(1-\lambda)}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda}(\frac{\int_0^{N_1} l_{1j} dj}{\int_0^{N_2} l_{2j} dj})^{1-\lambda} \\
&= (\frac{\epsilon}{1-\epsilon})^{\lambda}(\frac{Z_1}{Z_2})^{(2\delta+2\gamma-1)(1-\lambda)}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda}(\frac{L_1}{L_2})^{1-\lambda} \\
&= \frac{\epsilon}{1-\epsilon}(\frac{Z_1}{Z_2})^{(2\delta+2\gamma-1)(1-\lambda)}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda}
\end{aligned} \tag{27}$$

Substitute this expression into the inverse demand function 7 and then substitute the result into 26 to get

$$\begin{aligned}
P_{X_1} &= \epsilon X_1^{\epsilon-1} X_2^{1-\epsilon} \\
&= \epsilon(\frac{\epsilon}{1-\epsilon})^{(\epsilon-1)}(\frac{Z_1}{Z_2})^{(2\delta+2\gamma-1)(1-\lambda)(\epsilon-1)}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda(\epsilon-1)}
\end{aligned}$$

We thus have

$$\begin{aligned}
r_1 &= \alpha_1 \delta A_1 \frac{1-\lambda}{\lambda}(\frac{\lambda P_{X_1}}{A_1/\lambda})^{\frac{1}{1-\lambda}} Z_{1j}^{\delta+\gamma-1} Z_2^{1-(\delta+\gamma)} l_{1j} \\
&= \delta\alpha_1 A_1 \frac{1-\lambda}{\lambda}[\frac{\lambda^2\epsilon}{A_1}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{(\epsilon-1)}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{\frac{-\lambda(\epsilon-1)}{1-\lambda}}(\frac{Z_1}{Z_2})^{\Gamma-1} l_{1j}
\end{aligned} \tag{28}$$

where $\Gamma = \delta + \gamma + [2(\delta+\gamma)-1](\epsilon-1) \in (0,1)$. The restriction that $\Gamma \in (0,1)$ can be seen in two steps. First, Γ is positive:

$$\begin{aligned}
\delta + \gamma + [2(\delta+\gamma)-1](\epsilon-1) &= \delta + \gamma + (1-2(\delta+\gamma))(1-\epsilon) \\
&> 0
\end{aligned}$$

given that $\gamma > 0$, $\delta > 0$, and $0 < \delta + \gamma < 1$. Second, $\Gamma < 1$:

$$\begin{aligned}
\delta + \gamma + [2(\delta+\gamma)-1](\epsilon-1) &= 2(\delta+\gamma)\epsilon - (\delta+\gamma) - \epsilon + 1 \\
&= 1 - (\delta+\gamma)(1-\epsilon) - \epsilon[1-(\delta+\gamma)] \\
&< 1
\end{aligned}$$

given that $\delta + \gamma < 1$ and $\epsilon \in (0,1)$.

[1] Note that the monopoly firm still sets price as 22, even though it is facing a larger market size. That is because the price elasticity of the demand is constant.

In Section (2.4) we will see that zero entry cost induces entry if incipient profits arise. In equilibrium, distributed profit $\pi_{1j} = F_{1j} - R_{1j}$ is zero, i.e.,

$$\begin{aligned} 0 &= \pi_{1j} = F_{1j} - R_{1j} \\ &= G_{1j}\left(P_{G_{1j}} - A_1\right) - \theta_1 \frac{Z_1^2}{Z_2} - R_{1j} \\ &= A_1 \frac{1-\lambda}{\lambda} [\frac{\lambda^2 \epsilon}{A_1}]^{\frac{1}{1-\lambda}} (\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}} (\frac{P_{G_1}}{P_{G_2}})^{\frac{-\lambda(\epsilon-1)}{1-\lambda}} Z_1^{\Gamma} Z_2^{1-\Gamma} l_{1j} - \theta_1 \frac{Z_1^2}{Z_2} - R_{1j} \end{aligned}$$

We have $L_1 \equiv \int_0^{N_1} l_{1j}$. From 16 and $L_1 + L_2 = L$, we know that the total market size for industry 1, L_1, is fixed. Thus entry makes l_{1j} equal among firms in the industry. We also have $l_1 = L_1/N_1$.

We have obtained the results that all firms choose the same $R_{1j} \equiv R_1$ and the same output prices $P_{G_{1j}} = A_1/\lambda \equiv P_{G_1}$. Thus all firms choose the same levels of the two things they control, output price and R&D spending. Because all firms in the industry make the same choices, we henceforth drop the firm subscript.

2.2 Incumbents in Industry 2

Following the similar steps we get the results in industry 2.

$$CVH_2 = G_2(P_{G_2} - A_2) - \theta_2 \frac{Z_2^2}{Z_1} - R_2 + q_2(\alpha_2 I_2) \tag{29}$$

s.t. $G_2 = [\lambda P_{X_2}/P_{G_2}]^{\frac{1}{1-\lambda}} Z_2^{\delta+\gamma} Z_1^{1-(\delta+\gamma)} L_2/N_2$ and $\dot{Z}_2 = \alpha_2 R_2$. As before, $r_1 = r_2$, so $I_2 = R_2$.The first-order condition $\partial CVH_2/\partial P_{G2} = 0$ implies

$$P_{G_2} = \frac{A_2}{\lambda} \tag{30}$$

The first-order condition $\partial CVH_2/\partial R_2 = -1 + q_2\alpha_2$ implies

$$\begin{cases} R_2 = \infty & \text{if } \ 1/\alpha_2 > q_2 \\ R_2 > 0 & \text{if } \ 1/\alpha_2 = q_2 \\ R_2 = 0 & \text{if } \ 1/\alpha_2 < q_2 \end{cases}$$

The transversality condition is $lim_{t\to\infty} e^{-\int_t^{\tau} r(s)ds} q_{2j}(t) Z_{2j}(t) = 0$. Profit is

$$\begin{aligned} F_2 &= G_2(P_{G_2} - A_2) - \theta_2 \frac{Z_2^2}{Z_1} \\ &= A_2(\frac{1}{\lambda} - 1)\{\frac{\lambda P_{X_2}}{P_{G_2}}\}^{\frac{1}{1-\lambda}} (Z_{2j}^{\delta} Z_2^{\gamma} Z_1^{1-(\delta+\gamma)} l_2) - \theta_2 \frac{Z_2^2}{Z_1} \\ &= A_2 \frac{1-\lambda}{\lambda} [\frac{\lambda P_{X_2}}{A_2/\lambda}]^{\frac{1}{1-\lambda}} (Z_{2j}^{\delta} Z_2^{\gamma} Z_1^{1-(\delta+\gamma)} l_2) - \theta_2 \frac{Z_2^2}{Z_1} \end{aligned} \tag{31}$$

where

$$\begin{aligned} P_{X_2} &= (1-\epsilon)(\frac{X_1}{X_2})^{\epsilon} \\ &= (1-\epsilon)(\frac{\epsilon}{1-\epsilon})^{\epsilon} (\frac{Z_1}{Z_2})^{(2\delta+2\gamma-1)(1-\lambda)\epsilon} (\frac{P_{G_1}}{P_{G_2}})^{-\lambda\epsilon} \end{aligned}$$

We then obtain

$$\begin{aligned}\frac{\partial F_2}{\partial Z_{2j}} &= \delta A_2 \frac{1-\lambda}{\lambda}[\frac{\lambda P_{X_2}}{A_2/\lambda}]^{\frac{1}{1-\lambda}}(Z_2^{\delta+\gamma-1}Z_1^{1-(\delta+\gamma)})l_2 \\ &= \delta A_2 \frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{A_2}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{Z_1}{Z_2})^{-(1-2\delta-2\gamma)\epsilon}(\frac{P_{G_1}}{P_{G_2}})^{-\frac{\lambda\epsilon}{1-\lambda}}Z_2^{\delta-1}Z_1^{1-\delta}l_2\end{aligned} \tag{32}$$

and

$$\begin{aligned}r_2 &= \frac{\partial F_2}{\partial Z_2}\frac{1}{q_2} + \frac{\dot{q_2}}{q_2} = \frac{\partial F_2}{\partial Z_2}\frac{1}{q_2} + 0 = \frac{\partial F_2}{\partial Z_{2j}}\alpha_2 \\ &= \delta\alpha_2 A_2 \frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{A_2}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{-\frac{\lambda\epsilon}{1-\lambda}}(\frac{Z_1}{Z_2})^{\Gamma}l_2\end{aligned} \tag{33}$$

Because $1 > \Gamma > 0$, there are diminishing returns to Z_2.

We obtain all the same symmetry results as before. Thus firms within each industry are symmetric. However, in general $P_{G_1} = A_1/\lambda \neq A_2/\lambda = P_{G_2}$ and also $I_1 = R_1 \neq R_2 = I_2$ because in general $1/\alpha_1 \neq 1/\alpha_2$. Thus firms differ across the two industries, and the model is one of partially asymmetric equilibrium.

2.3 The Ratio X_1/X_2 Again

Using 19 and 16, we can write $\frac{X_1}{X_2}$ as

$$\begin{aligned}\frac{X_1}{X2} &= (\frac{\epsilon}{1-\epsilon})^{\lambda}(\frac{Z_1}{Z_2})^{(\delta+2\gamma-1)(1-\lambda)}(\frac{\int_0^{N_1} l_{1j}P_{G_{1j}}^{-\frac{\lambda}{1-\lambda}}Z_{1j}^{\delta}dj}{\int_0^{N_2} l_{2j}P_{G_{2j}}^{-\frac{\lambda}{1-\lambda}}Z_{2j}^{\delta}dj})^{1-\lambda} \\ &= (\frac{\epsilon}{1-\epsilon})^{\lambda}(\frac{Z_1}{Z_2})^{(\delta+2\gamma-1)(1-\lambda)}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda}(\frac{\int_0^{N_1} Z_{1j}^{\delta}l_{1j}dj}{\int_0^{N_2} Z_{2j}^{\delta}l_{2j}dj})^{1-\lambda}\end{aligned} \tag{34}$$

2.4 Zero Entry Cost, Zero Profit, and the Growth Rates of Z_1 and Z_2

Zero entry cost implies that the zero profit condition holds at all times (see the main text for more explanation). Distributed profit is $\Pi_i = F_i - R_i$, so the zero profit condition requires that $R_i = F_i$. Then using $\dot{Z}_1 = \alpha_1 R_1$ and $\dot{Z}_2 = \alpha_2 R_2$ with the zero profit condition, 24, and 31, we get the growth rates of Z_1 and Z_2:

$$\begin{aligned}g_1 &\equiv \frac{\dot{Z}_1}{Z_1} = \frac{\alpha_1 R_1}{Z_1} = \frac{\alpha_1 F_1}{Z_1} \\ &= \alpha_1\{A_1\frac{1-\lambda}{\lambda}[\frac{\lambda^2\epsilon}{A_1}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{\frac{-\lambda(\epsilon-1)}{1-\lambda}}\frac{L_1}{N_1}(\frac{Z_1}{Z_2})^{\Gamma-1} - \theta_1\frac{Z_1}{Z_2}\}\end{aligned} \tag{35}$$

$$\begin{aligned}g_2 &\equiv \frac{\dot{Z}_2}{Z_2} = \frac{\alpha_2 R_2}{Z_2} = \frac{\alpha_2 F_2}{Z_2} \\ &= \alpha_2\{A_2\frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{A_2}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{\frac{-\lambda\epsilon}{1-\lambda}}\frac{L_2}{N_2}(\frac{Z_2}{Z_1})^{-\Gamma} - \theta_2\frac{Z_2}{Z_1}\}\end{aligned} \tag{36}$$

3 General Equilibrium

3.1 Market Clearing Conditions

There are three markets that must clear simultaneously: labor, goods, and credit (or asset).

3.1.1 Labor Market

Labor supply is

$$L = L_1 + L_2 = \int_0^{N_1} l_{1j} dj + \int_0^{N_2} l_{2j} dj \tag{37}$$

Labor demand is given by equations 12 and 14, and from 16 we know

$$\frac{L_1}{L_2} = \frac{\epsilon}{1-\epsilon}$$

so

$$L_1 = \epsilon L \tag{38}$$

$$L_2 = (1-\epsilon) L \tag{39}$$

3.1.2 Goods Market

The resource constraint is

$$\begin{aligned} Y &= \int_0^{N_1} A_1 G_{1j} dj + \int_0^{N_1} R_{1j} dj + \int_0^{N_1} \phi_{1j} dj + \int_0^{N_2} A_2 G_{2j} dj \qquad (40) \\ &\quad + \int_0^{N_2} R_{2j} dj + \int_0^{N_2} \phi_{2j} dj + C \qquad (41) \end{aligned}$$

where $\phi_{1j} = \theta_1 Z_1^2 / Z_2,\ \phi_{2j} = \theta_2 Z_2^2 / Z_1$.

The factor payments are the following:

1. Final good producers pay $P_{X_1} X_1$ to the firms in the X_1 sector. Competition and Cobb-Douglas production guarantee that $P_{X_1} X_1 = \epsilon 1 \cdot Y$.
2. Similarly, final good producers pay $P_{X_2} X_2 = (1-\epsilon) 1 \cdot Y$ to the firms in the X_2 sector.
3. The firms in the X_1 sector pay $\int_0^{N_1} P_{G_{1_j}} G_{1j} dj = \lambda P_{X_1} X_1$ to the firms in the G_1 sector.
4. The firms in the X_1 sector pay $\int_0^{N_1} w l_{1j} dj = (1-\lambda) P_{X_1} X_1$ to labor.
5. The firms in the X_2 sector pay $\int_0^{N_2} P_{G_{2_j}} G_{2j} dj = \lambda P_{X_2} X_2$ to the firms in the G_2 sector.
6. The firms in the X_2 sector pay $\int_0^{N_2} w l_{2j} dj = (1-\lambda) P_{X_2} X_2$ to labor.

Summing these, we can write

$$\begin{aligned} 1 \cdot Y &= P_{X_1} X_1 + P_{X_2} X_2 \\ &= \int_0^{N_1} P_{G_{1_j}} G_{1j} dj + \int_0^{N_1} w l_{1j} dj + \int_0^{N_2} P_{G_{2_j}} G_{2j} dj + \int_0^{N_2} w l_{2j} dj \end{aligned} \tag{42}$$

The zero profit condition requires $P_{G_{1j}}G_{1j} = A_1G_{1j} + R_{1j} + \phi_{1j}$; and $P_{G_{2j}}G_{2j} = A_2G_{2j} + R_{2j} + \phi_{2j}$. So 42 can be written as

$$Y = \int_0^{N_1} A_1G_{1j}dj + \int_0^{N_1} R_{1j}dj + \int_0^{N_1} \phi_{1j}dj + \int_0^{N_2} A_1G_{2j}dj \tag{43}$$

$$+ \int_0^{N_2} R_{2j}dj + \int_0^{N_2} \phi_{2j}dj + wL \tag{44}$$

Comparing 40 and 43, we get

$$\begin{aligned} C &= wL \\ &= (1-\lambda)(P_{X_1}X_1 + P_{X_2}X_2) \\ &= (1-\lambda)Y \end{aligned}$$

Dividing by Y gives

$$\frac{wL}{Y} = \frac{C}{Y} = 1 - \lambda$$

which implies

$$\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{w}}{w} \tag{45}$$

because L and $1-\lambda$ are constant.

3.1.3 Credit Market (No Arbitrage Condition)

No arbitrage requires that all rates of return are equal:

$$r_1 = r_2 = r \tag{46}$$

so the right sides of 26 and 33 must be equal. If one return is higher, then all resources go to accumulating that quality. The numbers of firms N_1 and N_2 in each intermediate good industry jump up or down, which lowers or raises firm size l_1 and l_2. From 26 and 31, changes in firm size lower or raise the rates of return r_1 and r_2. Those changes continue until the arbitrage opportunity has been eliminated. See section 4 below for more discussion.

3.2 Derivations of the Growth Rates $\frac{\dot{G}_1}{G_1}$, $\frac{\dot{G}_2}{G_2}$, $\frac{\dot{X}_1}{X_1}$, $\frac{\dot{X}_2}{X_2}$,$\frac{\dot{Y}}{Y}$,$\frac{\dot{w}}{w}$, and $\frac{\dot{C}}{C}$

Substitute the inverse demand for X_15 and the expression 27 for X_1/X_2 into demand for G_1, 11 and rearrange terms to obtain

$$\begin{aligned} G_1 &= (\frac{\lambda\epsilon}{P_{G_1}})^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}}(\frac{P_{G1}}{P_{G2}})^{-\frac{\lambda(\epsilon-1)}{1-\lambda}}(\frac{Z_1}{Z_2})^{-(1-2(\delta+\gamma))(1-\epsilon)}(Z_1^{\delta+\gamma}Z_2^{1-(\delta+\gamma)}\frac{L_1}{N_1}) \\ &= \xi_1 Z_1^{[1-2(\delta+\gamma)](1-\epsilon)+(\delta+\gamma)}Z_2^{-[1-2(\delta+\gamma)](1-\epsilon)+1-(\delta+\gamma)}\frac{L_1}{N_1} \end{aligned}$$

where $\xi_1 = (\lambda\epsilon/P_{G_1})^{1/(1-\lambda)}[\epsilon/(1-\epsilon)]^{(\epsilon-1)/(1-\lambda)}(P_{G1}/P_{G2})^{-\lambda(\epsilon-1)/(1-\lambda)}$. The prices P_{G_1} and $P_{G_2}$are constant and equal to the unit cost times the monopolistic markup. The number of firms N_1 is a jumping variable. Labor in the first intermediate goods industry is the constant $L_1 = \epsilon L$. The only variables affecting the growth rate of G_1 are Z_1 and Z_2. For given L_1/N_1,

$$\frac{\dot{G}_1}{G_1} = \Gamma\frac{\dot{Z}_1}{Z_1} + (1-\Gamma)\frac{\dot{Z}_2}{Z_2} \tag{47}$$

where as before $\Gamma \equiv [1-2(\delta+\gamma)](1-\epsilon) + \delta = 1 - \epsilon - (\delta+\gamma) + 2(\delta+\gamma)\epsilon$. By a similar derivation for G_2, we obtain

$$G_2 = \xi_2 Z_1^{1-\epsilon-(\delta+\gamma)+2(\delta+\gamma)\epsilon}Z_2^{\epsilon+(\delta+\gamma)-2\epsilon(\delta+\gamma)}\frac{L_2}{N_2}$$

where $\xi_2 = (\lambda(1-\epsilon)/P_{G_2})^{1/(1-\lambda)}(\epsilon/(1-\epsilon))^{\epsilon/(1-\lambda)}(P_{G1}/P_{G2})^{-\lambda\epsilon/(1-\lambda)}$. The growth rate of G_2 is

$$\frac{\dot{G}_2}{G_2} = \Gamma\frac{\dot{Z}_1}{Z_1} + (1-\Gamma)\frac{\dot{Z}_2}{Z_2}$$

To obtain the growth rate of X_1, we must write the X_1 in terms of the quality levels Z_1 and Z_2. Plug the demand function for G_1 11 into 10 to get 17. Combine 27, 38 and 17 and rearrange terms to obtain

$$\begin{aligned} X_1 &= [\frac{\lambda\epsilon(\frac{X_1}{X_2})^{\epsilon-1}}{P_{G_1}}]^{\frac{\lambda}{1-\lambda}} Z_1^{\delta+\gamma} Z_2^{1-(\delta+\gamma)} L_1 \qquad (48)\\ &= \{\frac{\lambda\epsilon[\frac{\epsilon}{1-\epsilon}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda}(\frac{Z_1}{Z_2})^{-[1-2(\delta+\gamma)](1-\lambda)}]^{\epsilon-1}}{P_{G_1}}\}^{\frac{\lambda}{1-\lambda}} Z_1^{\delta+\gamma} Z_2^{1-(\delta+\gamma)} L_1 \\ &= \frac{(\lambda\epsilon)^{\frac{\lambda}{1-\lambda}}}{P_{G_1}^{\frac{\lambda}{1-\lambda}}}(\frac{\epsilon}{1-\epsilon})^{\frac{\lambda(\epsilon-1)}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{-\frac{\lambda^2(\epsilon-1)}{1-\lambda}}(\frac{Z_1}{Z_2})^{[1-2(\delta+\gamma)]\lambda(1-\epsilon)} Z_1^{\delta+\gamma} Z_2^{1-(\delta+\gamma)}\epsilon L \\ &= \zeta_1 Z_1^{\delta+\gamma+(1-2(\delta+\gamma))\lambda(1-\epsilon)} Z_2^{1-(\delta+\gamma)-[1-2(\delta+\gamma)]\lambda(1-\epsilon)}(\epsilon L) \end{aligned}$$

where $\zeta_1 = (\lambda\epsilon)^{\lambda/(1-\lambda)}(\epsilon/(1-\epsilon))^{\lambda(\epsilon-1)/(1-\lambda)} P_{G_1}^{-\lambda/(1-\lambda)}(P_{G1}/P_{G2})^{\lambda^2(\epsilon-1)/(1-\lambda)}$. We easily obtain the growth rate of X_1 from 48:

$$\frac{\dot{X}_1}{X_1} = \vartheta_1\frac{\dot{Z}_1}{Z_1} + (1-\vartheta_1)\frac{\dot{Z}_2}{Z_2} \qquad (49)$$

where $\vartheta_1 = \delta+\gamma+[1-2(\delta+\gamma)]\lambda(1-\epsilon)$.

The growth rate of X_2 is obtained by similar steps. Combine 34, 39 and 18 and rearrange terms to get

$$\begin{aligned} X_2 &= N_2[\frac{\lambda(1-\epsilon)(\frac{X_1}{X_2})^{\epsilon}}{P_{G_2}}]^{\frac{\lambda}{1-\lambda}}(Z_2^{\delta+\gamma} Z_1^{1-(\delta+\gamma)}\frac{L_2}{N_2}) \qquad (50)\\ &= \{\frac{\lambda(1-\epsilon)[\frac{\epsilon}{1-\epsilon}(\frac{P_{G_1}}{P_{G_2}})^{-\lambda}(\frac{Z_1}{Z_2})^{-[1-2(\delta+\gamma)](1-\lambda)}]^{\epsilon}}{P_{G_2}}\}^{\frac{\lambda}{1-\lambda}} Z_2^{\delta+\gamma} Z_1^{1-(\delta+\gamma)} L_2 \\ &= \frac{[\lambda(1-\epsilon)]^{\frac{\lambda}{1-\lambda}}}{P_{G_2}^{\frac{\lambda}{1-\lambda}}}(\frac{\epsilon}{1-\epsilon})^{\frac{\lambda\epsilon}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{-\frac{\lambda^2\epsilon}{1-\lambda}}(\frac{Z_1}{Z_2})^{-[1-2(\delta+\gamma)]\lambda\epsilon} Z_2^{\delta+\gamma} Z_1^{1-(\delta+\gamma)}(1-\epsilon)L \\ &= \zeta_2 Z_1^{1-(\delta+\gamma)-[1-2(\delta+\gamma)]\lambda\epsilon} Z_2^{\delta+\gamma+[1-2(\delta+\gamma)]\lambda\epsilon}(1-\epsilon)L \end{aligned}$$

where $\zeta_2 = [\lambda(1-\epsilon)]^{\lambda/(1-\lambda)}(\epsilon/(1-\epsilon))^{\lambda\epsilon/(1-\lambda)} P_{G_2}^{-\lambda/(1-\lambda)}(\frac{P_{G_2}}{P_{G_1}})^{\lambda^2\epsilon/(1-\lambda)}$. The growth rate of X_2 is

$$\frac{\dot{X}_2}{X_2} = \vartheta_2\frac{\dot{Z}_1}{Z_1} + (1-\vartheta_2)\frac{\dot{Z}_2}{Z_2} \qquad (51)$$

where $\vartheta_2 = 1-(\delta+\gamma)-[1-2(\delta+\gamma)]\lambda\epsilon$.

We turn now to the growth rate of Y. From 1 we have $Y = X_1^{\epsilon}X_2^{1-\epsilon}$. Combine 48, 50, and 1 to get

$$\begin{aligned} Y &= X_1^{\epsilon}X_2^{1-\epsilon} \\ &= [\zeta_1 Z_1^{(\delta+\gamma)+[1-2(\delta+\gamma)]\lambda(1-\epsilon)} Z_2^{1-(\delta+\gamma)-[1-2(\delta+\gamma)]\lambda(1-\epsilon)}(\epsilon L)]^{\epsilon} \\ &\quad \bullet[\zeta_2 Z_1^{1-(\delta+\gamma)-[1-2(\delta+\gamma)]\lambda\epsilon} Z_2^{\delta+\gamma+[1-2(\delta+\gamma)]\lambda\epsilon}(1-\epsilon)L]^{1-\epsilon} \\ &= (\zeta_1^{\epsilon}\zeta_2^{1-\epsilon})(\epsilon L)^{\epsilon}(1-\epsilon)^{1-\epsilon}L^{1-\epsilon} \\ &\quad \bullet Z_1^{(\delta+\gamma)\epsilon+[1-(\delta+\gamma)](1-\epsilon)+[1-2(\delta+\gamma)]\lambda(1-\epsilon)\epsilon-[1-2(\delta+\gamma)]\lambda(1-\epsilon)\epsilon} \\ &\quad \bullet Z_2^{[1-(\delta+\gamma)]\epsilon+(\delta+\gamma)(1-\epsilon)+[1-2(\delta+\gamma)]\lambda(1-\epsilon)\epsilon-[1-2(\delta+\gamma)]\lambda(1-\epsilon)\epsilon} \qquad (52)\\ &= \kappa Z_1^{\Gamma} Z_2^{1-\Gamma} L \end{aligned}$$

where

$$\begin{aligned}\kappa &= (\zeta_1^{\epsilon}\zeta_2^{1-\epsilon})(\epsilon)^{\epsilon}(1-\epsilon)^{1-\epsilon} \\ &= \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}}P_{G_1}^{\frac{-\lambda\epsilon}{1-\lambda}}P_{G_2}^{\frac{-\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\epsilon}(1-\epsilon)^{(1-\epsilon)}\end{aligned}$$

The growth rate of Y is

$$\frac{\dot{Y}}{Y} = \Gamma\frac{\dot{Z}_1}{Z_1} + (1-\Gamma)\frac{\dot{Z}_2}{Z_2}$$

Note also that $\Gamma = \epsilon\vartheta_1 + (1-\epsilon)\vartheta_2$, which connects the growth rate of Y to the growth rates of X_1 and X_2.

Finally, constant population means that consumption per person grows at the same rate as aggregate consumption:

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C}$$

So from 45 we get

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C} = \frac{\dot{w}}{w} = \frac{\dot{Y}}{Y} = \Gamma\frac{\dot{Z}_1}{Z_1} + (1-\Gamma)\frac{\dot{Z}_2}{Z_2} \tag{53}$$

3.3 Euler Equation

The household maximizes lifetime utility $U(t) = \int_t^{\infty} log(c)\, e^{-\rho t}$ s.t. $\dot{S} = rS + wL - cL$, where $c = C/L$ is consumption per capita, S is asset holdings, and r is the rate of return on assets. The current value Hamiltonian is

$$H = log(c_t) + \psi_t(rS + wL - cL)$$

where $\psi_t = e^{\rho t}\lambda_t$ is the current value costate variable. The necessary conditions are:

$$\begin{aligned}\dot{S} &= rS + wL - cL \\ \dot{\psi} &= \rho\psi_t - \frac{\partial H}{\partial S_t} = (\rho - r)\psi_t \\ \frac{\partial H}{\partial c_t} &= \frac{1}{c_t} - \psi_t = 0 \\ & S_0 \quad \text{given} \\ lim_{t\to\infty} e^{-\rho T}\psi_T S_T &= 0\end{aligned}$$

Use the Euler equation $1/c_t = \psi_t$ and $\dot{\psi} = (\rho - r)\psi_t$ to obtain

$$\begin{aligned}-\frac{\dot{c}_t}{c_t^2} &= (\rho - r)\frac{1}{c_t} \\ \Rightarrow \frac{\dot{c}_t}{c_t} &= (r - \rho)\end{aligned} \tag{54}$$

4 Balanced Growth Path and Transition Dynamics

We now derive the dynamic behavior of the economy.

4.1 Balanced Growth Path

We begin with the balanced growth path (BGP).

We have nine unknowns: $g_1,\ g_2,\ r_1,\ r_2,\ L_1,\ L_2,\ Z_1/Z_2,\ N_1$, and N_2. We determine them with the nine equations 28, 33, 35, 36, 38, 39, 46, 54, and the BGP requirement that all growth rates be constant:

$$\frac{d}{dt}\frac{\dot{c}}{c} = \frac{d}{dt}\frac{\dot{C}}{C} = \frac{d}{dt}\frac{\dot{w}}{w} = \frac{d}{dt}\frac{\dot{Y}}{Y} = \frac{d}{dt}g_1 = \frac{d}{dt}g_2 \equiv 0 \tag{55}$$

For the sake of notational ease, we write 28 and 33 in simplified forms:

$$\begin{aligned} r_1 &= \delta\alpha_1 A_1 \frac{1-\lambda}{\lambda}[\frac{\lambda^2\epsilon}{A_1}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{(\epsilon-1)}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{\frac{-\lambda(\epsilon-1)}{1-\lambda}}(\frac{Z_1}{Z_2})^{\Gamma-1}\frac{L_1}{N_1} \\ &= \delta f_1(\frac{L_1}{N_1},\frac{Z_1}{Z_2}) \end{aligned} \tag{56}$$

$$\begin{aligned} r_2 &= \delta\alpha_2 A_2 \frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{A_2}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{P_{G_1}}{P_{G_2}})^{-\frac{\lambda\epsilon}{1-\lambda}}(\frac{Z_1}{Z_2})^{\Gamma}\frac{L_2}{N_2} \\ &= \delta f_2(\frac{L_2}{N_2},\frac{Z_1}{Z_2}) \end{aligned} \tag{57}$$

where the functions f_1 and f_2 comprise all the terms after δ in the first lines of 56 and 57, respectively. We use these simplified versions to write simplified versions of 35 and 36 as well:

$$g_1 = f_1(\frac{L_1}{N_1},\frac{Z_1}{Z_2}) - \alpha_1\theta_1(\frac{Z_1}{Z_2}) \tag{58}$$

$$g_2 = f_2(\frac{L_2}{N_2},\frac{Z_1}{Z_2}) - \alpha_2\theta_2(\frac{Z_2}{Z_1}) \tag{59}$$

Rearrange 58 to get

$$f_1 = g_1 + \alpha_1\theta_1(\frac{Z_1}{Z_2}) \tag{60}$$

Plug 60 into 56:

$$r_1 = \delta[g_1 + \alpha_1\theta_1(\frac{Z_1}{Z_2})] \tag{61}$$

The no arbitrage condition 46 requires that all rates of return are equal at all times, and on the BGP all growth rates are constant. Therefore the growth rate for consumption per person, given by the Euler equation 54, requires that r_1 is a constant. That then implies through 61 that the ratio Z_1/Z_2 is constant, which in turn implies that g_1 equals g_2. From 53, we see that

$$\begin{aligned} \frac{\dot{c}}{c} &= \Gamma g_1 + (1-\Gamma)g_2 \\ &= \Gamma g_1 + (1-\Gamma)g_1 \\ &= g_1 \end{aligned}$$

Substituting this result into the Euler equation 54, we obtain $g_1 + \rho = \delta[g_1 + \alpha_1\theta_1 Z_1/Z_2]$, so

$$g_1^\star \equiv \frac{\dot{Z}_1}{Z_1} = \frac{\delta}{1-\delta}\alpha_1\theta_1(\frac{Z_1}{Z_2})^\star - \frac{1}{1-\delta}\rho \tag{62}$$

and

$$\begin{aligned} r_1^\star &= g_1^\star + \rho \\ &= \frac{\delta}{1-\delta}\alpha_1\theta_1(\frac{Z_1}{Z_2})^\star - \frac{1}{1-\delta}\rho + \rho \end{aligned} \tag{63}$$

We proceed similarly to get the BGP values of $g_2^\star$ and $r_2^\star$:

$$g_2^\star \equiv \frac{\dot{Z}_2}{Z_2} = \frac{\delta}{1-\delta}\alpha_2\theta_2(\frac{Z_2}{Z_1})^\star - \frac{1}{1-\delta}\rho \tag{64}$$

$$r_2^\star = g_2^\star + \rho = \frac{\delta}{1-\delta}\alpha_2\theta_2(\frac{Z_2}{Z_1})^\star - \frac{1}{1-\delta}\rho + \rho \tag{65}$$

Because $g_1 = g_2$, we set the right sides of 62 and 64 equal to each other and solve for Z_1/Z_2:

$$\begin{aligned} \frac{\delta}{1-\delta}\alpha_1\theta_1(\frac{Z_1}{Z_2})^\star - \frac{1}{1-\delta}\rho &= \frac{\delta}{1-\delta}\alpha_2\theta_2(\frac{Z_2}{Z_1})^\star - \frac{1}{1-\delta}\rho \\ \Rightarrow \alpha_1\theta_1(\frac{Z_1}{Z_2})^\star. &= \alpha_2\theta_2(\frac{Z_2}{Z_1})^\star \end{aligned}$$

$$\Rightarrow \alpha_1\theta_1(\frac{Z_1}{Z_2})^2 - \alpha_2\theta_2 = 0 \tag{66}$$

This quadratic has two roots

$$(\frac{Z_1}{Z_2})_1^\star = \pm\sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}}$$

The negative root is economically meaningless, so

$$(\frac{Z_1}{Z_2})^\star = \sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}} > 0 \tag{67}$$

If we plug 67 into 62 and 63, we get the balanced growth rate:

$$g^\star = g_1^\star = \frac{\delta}{1-\delta}\sqrt{\alpha_1\theta_1\alpha_2\theta_2} - \frac{1}{1-\delta}\rho \tag{68}$$

$$= g_2^\star \tag{69}$$

with the properties $\partial g^\star/\partial(\alpha_1\theta_1) > 0$ and $\partial g^\star/\partial(\alpha_2\theta_2) > 0$.

We immediately obtain the BGP solutions L_1^* and L_2^* from 38 and 39:

$$L_1^* = \epsilon L \tag{70}$$

$$L_2^* = (1-\epsilon)L \tag{71}$$

because ϵ and L are constant. The values L_1^* and L_2^* are the total amounts of labor employed in the X_1 and X_2 industries, respectively.

To get N_1, we begin by substituting the solutions for P_{G_1}, P_{G2}, and $(Z_1/Z_2)^*$ from 22, 30, and 67 into 28 to obtain

$$r_1^* = \alpha_1\delta A_1\frac{1-\lambda}{\lambda}[\frac{\lambda^2\epsilon}{A_1}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}}(\frac{A_1}{A_2})^{\frac{-\lambda(\epsilon-1)}{1-\lambda}}(\sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}})^{\Gamma-1}\frac{L_1}{N_1} \tag{72}$$

We then substitute 72 and 68 into 63 and rearrange terms to get

$$l_1^\star = \frac{L_1^*}{N_1^\star} = \frac{\epsilon L}{N_1^\star} = \frac{\frac{\delta}{1-\delta}\sqrt{\alpha_1\theta_1\alpha_2\theta_2} - \frac{\delta}{1-\delta}\rho}{\alpha_1\delta\frac{1-\lambda}{\lambda}[\lambda^2\epsilon]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}}}(\sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}})^{1-\Gamma}A_1^{\frac{\lambda\epsilon}{1-\lambda}}A_2^{\frac{\lambda(1-\epsilon)}{1-\lambda}} \tag{73}$$

$$\Rightarrow N_1^\star = \frac{\alpha_1\delta\frac{1-\lambda}{\lambda}[\lambda^2\epsilon]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}}}{\frac{\delta}{1-\delta}\sqrt{\alpha_1\theta_1\alpha_2\theta_2} - \frac{\delta}{1-\delta}\rho}(\sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}})^{\Gamma-1}A_1^{-\frac{\lambda\epsilon}{1-\lambda}}A_2^{-\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon L \tag{74}$$

which have the properties $\partial l_1^\star/\partial A_1 > 0$, $\partial l_1^\star/\partial A_2 > 0$, $\partial N_1^\star/\partial A_1 < 0$, and $\partial N_1^\star/\partial A_2 < 0$. Firm size L_1/N_1 is constant on the BGP. If L_1increases, N_1 jumps (zero entry cost) to keep the ratio constant. Consequently entry kills the scale effect at the aggregate level. We also see from 73 and 68 that the unit costs A_1 and A_2 do not enter the growth rate, though differences in unit costs do affect firm size. For example, an increase in unit cost A_1 causes an incipient decrease in profit. That has a negative effect on r_1 according to 28, and induces firms to leave instantaneously without any cost. As a result, firm size increases, which has a positive effect on r_1 according to 28. The two effects cancel, leaving r_1 unchanged. The reason that firm size l_1 is positively affected by unit cost A_2 in the other industry is that, when A_2 increases, P_{X_1} decreases according to inverse demand function for X_1, 7 and 19. Thus demand for G_{1j} decreases according to demand function of G_{1j}, 11. The result is a lower profit, inducing firms to leave.

By a similar analysis, we obtain the BGP values for l_2^* and N_2^*:

$$l_2^* = \frac{L_2^*}{N_2^\star} = \frac{(1-\epsilon)L}{N_2^\star} = \frac{\frac{\delta}{1-\delta}\sqrt{\alpha_1\theta_1\alpha_2\theta_2} - \frac{\delta}{1-\delta}\rho}{\alpha_2\delta\frac{1-\lambda}{\lambda}[\lambda^2(1-\epsilon)]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}}(\sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}})^{-\Gamma}A_1^{\frac{\lambda\epsilon}{1-\lambda}}A_2^{\frac{\lambda(1-\epsilon)}{1-\lambda}} \tag{75}$$

and

$$N_2^\star = \frac{\alpha_2\delta\frac{1-\lambda}{\lambda}[\lambda^2(1-\epsilon)]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}}{\frac{\delta}{1-\delta}\sqrt{\alpha_1\theta_1\alpha_2\theta_2} - \frac{\delta}{1-\delta}\rho}(\sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}})^{\Gamma}A_1^{-\frac{\lambda\epsilon}{1-\lambda}}A_2^{-\frac{\lambda(1-\epsilon)}{1-\lambda}}(1-\epsilon)L \tag{76}$$

On the BGP

$$\frac{\dot{c}}{c} = \frac{\dot{C}}{C} = \frac{\dot{w}}{w} = \frac{\dot{Y}}{Y} = \frac{\dot{Z}_1}{Z_1} = \frac{\dot{Z}_2}{Z_2} = r - \rho$$

4.2 Transition Dynamics

During the transition to the BGP, the growth rates of Z_1 and Z_2 generally will not be equal. Nonetheless, at all times

$$\frac{\dot{C}}{C} = \frac{\dot{w}}{w} = \frac{\dot{Y}}{Y} = \Gamma\frac{\dot{Z}_1}{Z_1} + (1-\Gamma)\frac{\dot{Z}_2}{Z_2}$$

from 53. From **??** and **??**, we get

$$\begin{aligned}
\frac{\dot{Z_1/Z_2}}{Z_1/Z_2} &= \frac{\dot{Z}_1}{Z_1} - \frac{\dot{Z}_2}{Z_2} \\
&= g_1 - g_2 \\
&= f_1(\frac{L_1}{N_1}, \frac{Z_1}{Z_2}) - \alpha_1\theta_1(\frac{Z_1}{Z_2}) - [f_2(\frac{L_2}{N_2}, \frac{Z_1}{Z_2}) - \alpha_2\theta_2(\frac{Z_2}{Z_1})] \\
&= f_1(\frac{L_1}{N_1}, \frac{Z_1}{Z_2}) - f_2(\frac{L_2}{N_2}, \frac{Z_1}{Z_2}) + \alpha_2\theta_2(\frac{Z_2}{Z_1}) - \alpha_1\theta_1(\frac{Z_1}{Z_2})
\end{aligned}$$

The no arbitrage condition 46 together with , 56 and 57 implies that

$$f_1(L_1/\!\!/N_1, Z_1/Z_2) = f_2(L_2/\!\!/N_2, Z_1/Z_2)$$

so

$$\frac{\dot{Z_1/Z_2}}{Z_1/Z_2} = \alpha_2\theta_2(\frac{Z_2}{Z_1}) - \alpha_1\theta_1(\frac{Z_1}{Z_2})$$

Multiply both sides by Z_1/Z_2 to get

$$(\dot{Z_1/Z_2}) = -\alpha_1\theta_1(\frac{Z_1}{Z_2})^2 + \alpha_2\theta_2$$

The two steady state values of Z_1/Z_2 are

$$\begin{aligned} \sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}} &> 0 \\ -\sqrt{\frac{\alpha_2\theta_2}{\alpha_1\theta_1}} &< 0 \end{aligned}$$

The steady state $\sqrt{\alpha_2\theta_2/\alpha_1\theta_1}$ is stable because $\dot{(Z_1/Z_2)} \gtreqless 0$ as $Z_1/Z_2 \lesseqgtr \sqrt{\alpha_2\theta_2/\alpha_1\theta_1}$ for all $Z_1/Z_2 > -\sqrt{\alpha_2\theta_2/\alpha_1\theta_1}$. The steady state $-\sqrt{\alpha_2\theta_2/\alpha_1\theta_1}$ is unstable because $\dot{(Z_1/Z_2)} \gtreqless 0$ as $Z_1/Z_2 \gtreqless -\sqrt{\alpha_2\theta_2/\alpha_1\theta_1}$. Thus for all non-negative values (which are the only economically meaningful values), Z_1/Z_2 converges to $\sqrt{\alpha_2\theta_2/\alpha_1\theta_1}$.

5 Open Economy Under Complete Specialization

5.1 Basic Setup

The final good production function in the home country is

$$Y_H = X_{H1}^{\epsilon} X_{H2}^{1-\epsilon}$$

and the production function for processed good 1 is

$$\begin{aligned} X_{H1} &= \int_0^{N_{H1}} (G_{H1j} - G_{H1j}^E)^{\lambda} [Z_{H1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1H}]^{1-\lambda} dj \\ &\quad + \int_0^{N_{F1}} (G_{F1j}^I)^{\lambda} [Z_{F1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1F}]^{1-\lambda} dj \end{aligned}$$

where G_H^E is the amount of G_H produced in the home country and exported, G_F^I is the amount of good G_F produced abroad and imported, l_{H1H} and l_{H1F} are the amounts of labor in the home country making X_{H1} using G_H andG_F, respectively, and $\widehat{Z_{H2}}$ is the quality spillover whose value depends on the goods being used:

$$\widehat{Z_{H2}} = \begin{cases} Z_{H2} & \text{if industry 2 chooses domestic good} \\ Z_{H2}^{\eta} Z_{F2}^{1-\eta} & \text{if industry 2 chooses both goods} \\ Z_{F2} & \text{if industry 2 chooses foreign good} \end{cases}$$

By following similar steps to those in Section 1 to maximize the profit of processed-good producer X_{H1} given the prices of intermediate goods and wage, we get the demand function for the domestic good, $(G_{H1j} - G_{H1j}^E)$, and the foreign good, G_{F1j}^I. Set the final good of home country as numeraire:

$$P_{Y_H} \equiv 1$$

Denote the price of the domestically produced intermediate good as $P_{G_{H1}}$ and price of the foreign good as $P_{G_{F1}}$. Maximizing profit of the processed good firm, given the prices of intermediate goods and wages, we get

$$P_{X_{H1}} [\lambda (G_{H1j} - G_{H1j}^E)^{\lambda-1} (Z_{H1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1H})^{1-\lambda}] = P_{G_{H1}}$$

so the demand function for domestic good is

$$G_{H1j} - G_{H1j}^E = [\frac{\lambda P_{X_{H1}}}{P_{G_{H1H}}}]^{\frac{1}{1-\lambda}} Z_{H1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1H} \tag{77}$$

Similarly, we get the demand for foreign good:

$$G^I_{F1j} = [\frac{\lambda P_{X_{H1}}}{P_{G_{H1F}}}]^{\frac{1}{1-\lambda}} Z_{F1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1F} \tag{78}$$

The demands for workers are,

$$l_{H1H} = (P_{X_{H1}} \frac{1-\lambda}{w})^{\frac{1}{\lambda}} (G_{H1j} - G^E_{H1j}) (Z^{\delta}_{H1j} Z^{\gamma}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)})^{\frac{1-\lambda}{\lambda}} \tag{79}$$

$$l_{H1F} = (P_{X_{H1}} \frac{1-\lambda}{w})^{\frac{1}{\lambda}} G^I_{F1j} (Z^{\delta}_{F1j} Z^{\gamma}_{F1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)})^{\frac{1-\lambda}{\lambda}} \tag{80}$$

Substitute the demand for labor 80 into 78 and note that the $G'_i s$ on both sides cancel, giving

$$\begin{aligned} G^I_{F1j} &= [\frac{\lambda P_{X_{H1}}}{P_{G_{H1F}}}]^{\frac{1}{1-\lambda}} Z_{F1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1H} \\ &= [\frac{\lambda P_{X_{H1}}}{P_{G_{H1F}}}]^{\frac{1}{1-\lambda}} Z_{F1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} (P_{X_{H1}} \frac{1-\lambda}{w})^{\frac{1}{\lambda}} G^I_{F1j} (Z^{\delta}_{F1j} Z^{\gamma}_{F1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)})^{\frac{1-\lambda}{\lambda}} \\ \Rightarrow w &= P_{X_{H1}} [\lambda P_{X_{H1}}]^{\frac{\lambda}{1-\lambda}} (\frac{1}{P_{G_{H1F}}})^{\frac{\lambda}{1-\lambda}} (1-\lambda) Z_{F1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} \end{aligned} \tag{81}$$

where w is the wage if the processed good firms use G^I_{F1j}. Similarly, substituting 79 into 77 gives

$$w = P_{X_{H1}} [\lambda P_{X_{H1}}]^{\frac{\lambda}{1-\lambda}} (\frac{1}{P_{G_{H1H}}})^{\frac{\lambda}{1-\lambda}} (1-\lambda) Z_{H1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} \tag{82}$$

Rearranging 81 and 82, we get

$$\begin{aligned} w_{H1H} &= \Upsilon (\frac{Z_{H1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{P_{G_{H1}}})^{\frac{\lambda}{1-\lambda}} \\ w_{H1F} &= \Upsilon (\frac{Z_{F1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{P_{G_{F1}}})^{\frac{\lambda}{1-\lambda}} \end{aligned} \tag{83}$$

where $\Upsilon \equiv P_{X_{H1}} (1-\lambda) [\lambda P_{X_{H1}}]^{\frac{\lambda}{1-\lambda}} (\widehat{Z_{H2}})^{1-(\delta+\gamma)}$.

From 83, given the spillover level $\widehat{Z_{H2}}$, the processed good producers in industry 1 choose the intermediate good 1 that gives the worker a higher marginal product. Processed-good producers X_2 do the same: given the spillover from industry 1, which is $\widehat{Z_{H1}}$, firms choose the intermediate good 2 which give the worker who are using them a higher marginal product. When producers of X_2 make decisions about G_2, what they care about is the quality-adjusted price. They ignore the externality they bring to industry 1, which is $\widehat{Z_{H2}}$. That externality is discussed in more detail in section (5.4).

Note that the intermediate good firms always set the prices $P_{G_{H1}}$ and $P_{G_{F1}}$ equal to unit cost multiplied by a markup[2]. The prices have nothing to do with the individual firm's market size because the demand function is constant elasticity. See Microeconomic Analysis (Third Edition) by Hal R. Varian, Section 14.1.

Recall that in the closed economy, we used this way to prove symmetry among firms the same industry. The wage is $W_i = (1-\lambda)(\lambda/P_i)^{\lambda/(1-\lambda)} Z_i^{\delta} Z^{1-\delta}$. Every firm in the industry sets the price at A_i/λ, so the only possible difference in the wage is the quality level, Z_i. If a firm has a higher quality, $Z_i > Z_j$, then $W_i > W_j$ and that firm will take over the whole market, become a monopoly, and earn a positive profit. Because entry cost is zero, new firms enter instantaneously with the average quality of the incumbent (by assumption), so the equilibrium is that all firms in the same industry have the same quality level.

[2] P_{G_2} needs to be rescaled by the final good price of foreign country.

5.2 Define the Trade Pattern

International trade takes place if each country has a lower (or equal) quality-adjusted price for one intermediate good. For concreteness and with no loss of generality, we assume throughout that the home country has a lower quality-adjusted price for G_1 goods and the foreign country has a lower quality-adjusted price for G_2 goods:

$$\frac{\mathbf{P_{G_{H1}}}}{\mathbf{Z_{H1}}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \leq \frac{\mathbf{P_{G_{F1}}}}{\mathbf{Z_{F1}}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \quad \textbf{and} \quad \frac{\mathbf{P_{G_{H2}}}}{\mathbf{Z_{H2}}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \geq \frac{\mathbf{P_{G_{F2}}}}{\mathbf{Z_{F2}}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \tag{84}$$

The numeraire is the final good in Home, so the price of the final good in Foreign is P_{Y_F}. Thus above condition becomes

$$\frac{A_{H1}/\lambda}{Z_{H1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \leq \frac{P_{Y_F}A_{F1}/\lambda}{Z_{F1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \quad \textbf{and} \quad \frac{A_{H2}/\lambda}{Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \geq \frac{P_{Y_F}A_{F2}/\lambda}{Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}$$

$$\begin{aligned} \Leftrightarrow \quad & \frac{A_{H2}}{A_{F2}}\left(\frac{Z_{F2}}{Z_{H2}}\right)^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} \geq P_{Y_F} \geq \frac{A_{H1}}{A_{F1}}\left(\frac{Z_{F1}}{Z_{H1}}\right)^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} \\ \Rightarrow \quad & \frac{A_{H2}}{A_{F2}}\left(\frac{Z_{F2}}{Z_{H2}}\right)^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} \geq \frac{A_{H1}}{A_{F1}}\left(\frac{Z_{F1}}{Z_{H1}}\right)^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} \end{aligned} \tag{85}$$

When strict inequality holds, each economy is in a state of complete specialization, producing only one class of intermediate good and importing the other from its trading partner. In this section, we discuss complete specialization. We discuss incomplete specialization in section (6).

5.2.1 Home Country : Final Good and Processed Goods

Under complete specialization, Home's production function for processed good X_1 simplifies to

$$X_{H1} = \int_0^{N_{H1}} (G_{H1j} - G^E_{H1j})^{\lambda} [Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} l_{H1}]^{1-\lambda}\, dj$$

because only Home produces intermediate goods G_{1j}. The amount of intermediate good 1 used by Home is $G_{H1j} - G^E_{H1j}$. The amount exported, G^E_{H1j}, equals the demand for G_1-type goods from Foreign. Home does import G_2-type goods from Foreign for use in Home's X_2 industry. The spillover in the X_1 industry associated with those goods is Z_{F2}.

Home's production function for processed good X_2 is

$$X_{H2} = \int_0^{N_{F2}} (G^I_{H2j})^{\lambda} [Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2}]^{1-\lambda}\, dj]^{1-\lambda}$$

Home uses only G_2-type goods imported from Foreign. The number of varieties of such goods is N_{F2}.

We already have proven symmetry for firms inside a given industry, and that proof carries over here. Consequently, we drop the subscript j. Following exactly the same steps as in section (1), we get the demand functions

$$G_{H1} - G^E_{H1} = \left[\frac{\lambda\epsilon(\frac{X_{H1}}{X_{H2}})^{\epsilon-1}}{P_{G_{H1}}}\right]^{\frac{1}{1-\lambda}} Z^{\delta}_{H1j} Z^{\gamma}_{H1} (Z_{F2})^{1-(\delta+\gamma)} l_{H1} \tag{86}$$

$$G^I_{H2} = \left[\frac{\lambda(1-\epsilon)(\frac{X_{H1}}{X_{H2}})^{\epsilon}}{P_{G_{F2}}}\right]^{\frac{1}{1-\lambda}} Z^{\delta}_{F2j} Z^{\gamma}_{F2} (Z_{H1})^{1-(\delta+\gamma)} l_{H2} \tag{87}$$

The trade balance condition is

$$N_{H1} P_{G_{H1}} G^E_{H1} = N_{F2} P_{G_{F2}} G^I_{H2}$$

5.2.2 Foreign Country: Final Good and Processed Goods

The final good production function in Foreign is

$$Y_F = X_{F1}^{\epsilon} X_{F2}^{1-\epsilon} \tag{88}$$

and the processed goods production functions are

$$X_{F1j} = \int_0^{N_{H1}} (G_{F1j}^I)^{\lambda} [Z_{H1j}^{\delta} Z_{H1}^{\gamma} Z_{F2}^{1-(\delta+\gamma)} l_{F1}]^{1-\lambda} \, dj \tag{89}$$

where the number of varieties of G_1 available in Foreign is N_{H1}, which is decided by Home, and

$$X_{F2j} = \int_0^{N_{F2}} (G_{F_{2j}} - G_{F2j}^E)^{\lambda} [Z_{F2j}^{\delta} Z_{F2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} l_{F2}]^{1-\lambda} \, dj \tag{90}$$

Symmetry holds in Foreign by the same arguments as in Home, so again we drop the subscript j. Final good producers' profit in Foreign is, in units of the numeraire (Home's final good Y_H)

$$\pi_{Y_F} = P_{Y_F} Y_F - P_{X_{F2}} X_{F2} - P_{X_{F2}} X_{F2}$$

The inverse demand functions are

$$\begin{aligned} P_{X_{F1}} &= P_{Y_F} \epsilon X_{F1}^{\epsilon-1} X_{F2}^{1-\epsilon} \\ P_{X_{F2}} &= P_{Y_F} (1-\epsilon) X_{F1}^{\epsilon} X_{F2}^{-\epsilon} \end{aligned}$$

Following similar steps as in Home, we get the demand functions of intermediate goods:

$$G_{F1j}^I = [\frac{\lambda\epsilon(\frac{X_{F1}}{X_{F2}})^{\epsilon-1} P_{Y_F}}{P_{G_{H1}}}]^{\frac{1}{1-\lambda}} Z_{H1j}^{\delta} Z_{H1}^{\gamma} Z_{F2}^{1-(\delta+\gamma)} l_{F1} \tag{91}$$

$$G_{F_{2j}} - G_{F2j}^E = [\frac{\lambda(1-\epsilon)(\frac{X_{F1}}{X_{F2}})^{\epsilon} P_{Y_F}}{P_{G_{F2}}}]^{\frac{1}{1-\lambda}} Z_{F2j}^{\delta} Z_{F2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} l_{H2} \tag{92}$$

The trade balance condition is

$$N_{H1} P_{G_{H1}} G_{H1}^E = N_{F2} P_{G_{F2}} G_{H2}^I \tag{93}$$

We also have the market equilibrium conditions

$$\begin{aligned} G_{F1}^I &= G_{H1}^E \\ G_{F2}^E &= G_{H2}^I \end{aligned}$$

Following exactly the same steps as in section (1) and (2), we also get

$$\frac{X_{H1}}{X_{H2}} = \frac{X_{F1}}{X_{F2}} = \frac{\epsilon}{1-\epsilon} (\frac{P_{G_{H1}}}{P_{G_{F2}}})^{-\lambda} (\frac{Z_{H1}}{Z_{F2}})^{-[1-2(\delta+\gamma)](1-\lambda)}$$

Adding 86 to 91, we get the total demand for G_{H1}:

$$\begin{aligned} G_{H1j} &= (G_{H1j} - G_{H1j}^E) + G_{F1j}^I \\ &= (G_{H1j} - G_{H1j}^E) + G_{H1j}^E \\ &= [\frac{\lambda\epsilon(\frac{X_{H1}}{X_{H2}})^{\epsilon-1}}{P_{G_{H1}}}]^{\frac{1}{1-\lambda}} Z_{H1j}^{\delta} Z_{H1}^{\gamma} Z_{F2}^{1-(\delta+\gamma)} \frac{L_{H1} + P_{Y_F}^{\frac{1}{1-\lambda}} L_{F1}}{N_{H1}} \end{aligned} \tag{94}$$

Adding 87 to 92, we get the total demand for G_{F2}:

$$G_{F2j} = [\frac{\lambda\epsilon(\frac{X_{F1}}{X_{F2}})^{\epsilon}}{P_{G_{F2}}}]^{\frac{1}{1-\lambda}} Z_{F2j}^{\delta} Z_{F2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} \frac{L_{H2} + P_{Y_F}^{\frac{1}{1-\lambda}} L_{F2}}{N_{F2}} \tag{95}$$

We derive the solution for P_{Y_F} in section (5.3) below.

Comparing 94 and 95 with the closed economy demand 11 and 13, we see that each firm now not only faces the domestic market but also an international market. In first-generation trade models, a higher population means a higher market size and hence induces firms to do R&D. Here, in contrast, the individual firm faces not the total population size but rather the individual market size, which is the population size divided by the number of firms. The number of firms changes endogenously, so an increase in effective population due to trade does not cause a higher growth rate. Any effect of trade on growth in this model does not arise through a scale effect.

5.2.3 Intermediate Goods Firms

Using the demand functions 94 and 95, along with the R&D production functions $\dot{Z_{H1}} = \alpha_{H1} R_{H1}$ and $\dot{Z_{F2}} = \alpha_{F2} R_{F2}$, we follow exactly the same steps as in section (2) to obtain the following.

1. Home Country:

Home intermediate producers set their price as a markup over unit cost:

$$P_{G_{H1}} = A_{H1}/\lambda$$

The marginal revenue product of their R&D is

$$\frac{\partial F_{H1}}{\partial Z_{H1}} = \delta A_{H1} \frac{1-\lambda}{\lambda} [\frac{\lambda^2 \epsilon}{A_{H1}}]^{\frac{1}{1-\lambda}} (\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}} (\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{-\lambda(\epsilon-1)}{1-\lambda}} (\frac{Z_{H1}}{Z_{F2}})^{\Gamma-1} \frac{L_{H1} + P_{Y_F}^{\frac{1}{1-\lambda}} L_{F1}}{N_{H1}}$$

The rate of return to Home R&D is

$$r_{H1} = \frac{\partial F_{H1}}{\partial Z_{H1}} \alpha_{H1} \tag{96}$$

The rate of growth of the Home intermediate good's quality Z_{H1} is

$$\begin{aligned} g_{H1} &\equiv \frac{\dot{Z}_{H1}}{Z_{H1}} = \frac{\alpha_{H1} R_{H1}}{Z_{H1}} = \frac{\alpha_{H1} F_{H1}}{Z_{H1}} \\ &= \alpha_{H1} \{ A_{H1} \frac{1-\lambda}{\lambda} [\frac{\lambda^2 \epsilon}{A}]^{\frac{1}{1-\lambda}} (\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}} (\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{-\lambda(\epsilon-1)}{1-\lambda}} (\frac{Z_{H1}}{Z_{F2}})^{\Gamma-1} \frac{L_{H1} + P_{Y_F}^{\frac{1}{1-\lambda}} L_{F1}}{N_{H1}} - \theta_{F1} \frac{Z_{H1}}{Z_{F2}} \} \end{aligned}$$

2. Foreign Country

Foreign intermediate producers also set their price as a markup over unit cost:

$$P_{G_{F2}} = P_{Y_F} A_{F2}/\lambda$$

The marginal revenue product of Foreign R&D is[3]

$$\frac{\partial F_{F2}}{\partial Z_{F2}} = P_{F_2} \delta A_{F2} \frac{1-\lambda}{\lambda} [\frac{\lambda^2 (1-\epsilon)}{A_{F2}}]^{\frac{1}{1-\lambda}} (\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}} (\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{-\lambda\epsilon}{1-\lambda}} (\frac{Z_{F2}}{Z_{H1}})^{-\Gamma} (l_{H2} P_{Y_F}^{\frac{-1}{1-\lambda}} + l_{F2})$$

and the rate of return to R&D is

$$r_{F2} = \frac{\partial F_{F2}}{\partial Z_{F2}} / q_{F2};\ q_{F2} = \frac{P_{Y_2}}{\alpha_{F2}} \tag{97}$$

[3] $l_{H2} P_{Y_F}^{\frac{-1}{1-\lambda}} + l_{F2} \equiv P_{Y_F}^{\frac{-1}{1-\lambda}} \frac{L_{H1}}{N_{F2}} + \frac{L_{F2}}{N_{F2}}$

The rate of growth of Foreign quality Z_{F2} is

$$\begin{aligned} g_{F2} &\equiv \frac{\dot{Z}_{F2}}{Z_{F2}} = \frac{\alpha_{F2}R_{F2}}{Z_{F2}} = \frac{\alpha_{F2}F_{F2}/P_{Y_F}}{Z_{F2}} \\ &= \alpha_{F2}\{A_{F2}\frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{A_{F2}}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{-\lambda\epsilon}{1-\lambda}}(l_{H2}P_{Y_F}^{\frac{-1}{1-\lambda}}+l_{F2})(\frac{Z_{F2}}{Z_{H1}})^{-\Gamma}-\theta_{F2}\frac{Z_{F2}}{Z_{H1}}\} \end{aligned}$$

Note that in r_{F2}, the market size by each firm we express as $l_{H2}P_{Y_F}^{\frac{-1}{1-\lambda}}+l_{F2}$, but in 95, it was $l_{H2}+l_{F2}P_{Y_F}^{\frac{1}{1-\lambda}}$[4]. The difference arises from using $P_{Y_{F2}}$to convert to units of the Home final good. The details of the calculations follow.the same steps as in section (2.2). The current value Hamiltonian

$$CVH_{F2} = G_{F2}(P_{G_{F2}} - P_{Y_{F2}}A_{F2}) - P_{Y_{F2}}\theta_{F2}\frac{Z_{F2}^2}{Z_{H1}} - P_{Y_F}R_{F2} + q_{F2}(\alpha_{F2}R_{F2})$$

is maximized s.t. the demand function 95 and the R&D production function $\dot{Z}_{F2} = \alpha_{F2}R_{F2}$:

$$\frac{\partial CVH_{F2}}{\partial P_{GF2}} = 0 \Rightarrow P_{G_{F2}} = P_{Y_F}A_{F2}/\lambda$$

$$\frac{\partial CVH_{F2}}{\partial R_{F2}} = -P_{Y_F} + q_{2F}\alpha_{F2} \Rightarrow Interior\ solution: \ \ q_{2F} = \frac{P_{Y_2}}{\alpha_{F2}}$$

$$F_{F2} = G_{F2}(P_{G_{F2}} - P_{Y_F}A_{F2}) - \theta_{F2}\frac{Z_{H1}+Z_{F2}}{2}$$

Substituting 95 into the last equation gives

$$\begin{aligned} F_{F2j} &= P_{Y_F}A_{F2}(\frac{1}{\lambda}-1)\{\frac{\lambda(1-\epsilon)[\frac{\epsilon}{1-\epsilon}(\frac{P_{GH1}}{P_{GF2}})^{-\lambda}(\frac{Z_{H1}}{Z_{F2}})^{-[1-2(\delta+\gamma)](1-\lambda)}]^{\epsilon}}{P_{G_{F2}}}\}^{\frac{1}{1-\lambda}} \\ &\quad \bullet[Z_{F2j}^{\delta}Z_{F2}^{\gamma}Z_{H1}^{1-(\delta+\gamma)}(l_{H2}+P_{Y_F}^{\frac{1}{1-\lambda}}l_{F2})] - P_{Y_F}\theta_{F2}\frac{Z_{F2}^2}{Z_{H1}} \\ &= P_{Y_F}\{A_{F2}\frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{P_{Y_F}A_{F2}}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{P_{GH1}}{P_{GF2}})^{-\frac{\lambda\epsilon}{1-\lambda}}(\frac{Z_{H1j}}{Z_{F2j}})^{-[1-2(\delta+\gamma)](\epsilon-1)} \\ &\quad \bullet[Z_{F2j}^{\delta}Z_{F2}^{\gamma}Z_{H1j}^{1-(\delta+\gamma)}(l_{H2}+P_{Y_F}^{\frac{1}{1-\lambda}}l_{F2})] - \theta_{F2}\frac{Z_{F2}^2}{Z_{H1}}\} \\ &= P_{Y_F}\{A_{F2}\frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{A_{F2}}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{P_{GH1}}{P_{GF2}})^{-\frac{\lambda\epsilon}{1-\lambda}}(\frac{Z_{H1j}}{Z_{F2j}})^{-[1-2(\delta+\gamma)](\epsilon-1)} \\ &\quad \bullet[Z_{F2j}^{\delta}Z_{F2}^{\gamma}Z_{H1j}^{1-(\delta+\gamma)}P_{Y_F}^{\frac{-1}{1-\lambda}}(l_{H2}+P_{Y_F}^{\frac{1}{1-\lambda}}l_{F2})] - \theta_{F2}\frac{Z_{F2}^2}{Z_{H1}}\} \\ &= P_{Y_F}\{A_{F2}\frac{1-\lambda}{\lambda}[\frac{\lambda^2(1-\epsilon)}{A_{F2}}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon}{1-\lambda}}(\frac{P_{GH1}}{P_{GF2}})^{-\frac{\lambda\epsilon}{1-\lambda}}(\frac{Z_{H1j}}{Z_{F2j}})^{-[1-2(\delta+\gamma)](\epsilon-1)} \\ &\quad \bullet[Z_{F2j}^{\delta}Z_{F2}^{\gamma}Z_{H1j}^{1-(\delta+\gamma)}(l_{H2}P_{Y_F}^{\frac{-1}{1-\lambda}}+l_{F2})] - \theta_{F2}\frac{Z_{F2}^2}{Z_{H1}}\} \end{aligned}$$

The total market size is the sum of the domestic market and the foreign market, $l_{H2}+P_{Y_F}^{\frac{1}{1-\lambda}}l_{F2}$. The price is $P_{Y_F}A_{F2}(\lambda\epsilon-1)/\lambda(\epsilon-1)$. After converting to units of Home final output, these expressions become $l_{H2}P_{Y_F}^{\frac{-1}{1-\lambda}}+l_{F2}$ and A_{F2}/λ.

[4] $l_{H2}+l_{F2}P_{Y_F}^{\frac{1}{1-\lambda}} \equiv \frac{L_{H2}+P_{Y_F}^{\frac{1}{1-\lambda}}L_{F2}}{N_{F2}}$

5.3 Trade Balance Condition under Complete Specialization: $P_{Y_F} = [\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda}$

As we mentioned in section (2.1), the final good producer in Home pays compensation $(1-\epsilon)\lambda Y_H$ to the intermediate producers of industry 2, which now are Foreign firms. Similarly, the final good producer in Foreign pays compensation $\epsilon\lambda P_{Y_F} Y_F$, measured with the final good price from Home, to the intermediate producers of industry 1, which are Home firms. Trade balance requires

$$(1-\epsilon)\lambda Y_H = \epsilon\lambda P_{Y_F} Y_F \tag{98}$$

The final good production of Home is

$$Y_H = \kappa_H Z_{H1}^{\Gamma} Z_{F2}^{1-\Gamma} L_H \tag{99}$$

where

$$\kappa_H = \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}} P_{G_{H1}}^{-\frac{\lambda\epsilon}{1-\lambda}} P_{G_{H2}}^{\frac{-\lambda(1-\epsilon)}{1-\lambda}} \epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon}$$

Final good production of Foreign is scaled by the relative final good price, P_{Y_F}. Following the same step as in the derivation of 52, we combine 91, 92, 89, and 90 with the production function of foreign final good 88 to get

$$\begin{aligned} Y_F &= X_{F1}^{\epsilon} X_{F2}^{1-\epsilon} \\ &= [\zeta_{F1} P_{Y_F}^{\frac{\lambda}{1-\lambda}} Z_{H1}^{(\delta+\gamma)+[1-2(\delta+\gamma)]\lambda(1-\epsilon)} Z_{F2}^{1-(\delta+\gamma)-[1-2(\delta+\gamma)]\lambda(1-\epsilon)} (\epsilon L_F)]^{\epsilon} \\ &\quad \bullet [\zeta_{F2} P_{Y_F}^{\frac{\lambda}{1-\lambda}} Z_{H1}^{1-(\delta+\gamma)-[1-2(\delta+\gamma)]\lambda\epsilon} Z_{F2}^{(\delta+\gamma)+[1-2(\delta+\gamma)]\lambda\epsilon} (1-\epsilon)L_F]^{1-\epsilon} \\ &= \kappa_F Z_{H1}^{\Gamma} Z_{F2}^{1-\Gamma} (L_F P_{Y_F}^{\frac{\lambda}{1-\lambda}}) \end{aligned} \tag{100}$$

where

$$\begin{aligned} \kappa_F &= \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}} P_{G_{H1}}^{-\frac{\lambda\epsilon}{1-\lambda}} P_{G_{H2}}^{\frac{-\lambda(1-\epsilon)}{1-\lambda}} \epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon} \\ &= \kappa_H \end{aligned}$$

and $P_{G_{F2}} = P_{Y_F} A_{F2}/\lambda$. Substitute 100 and 99 into 98 to get

$$(1-\epsilon)L_H = \epsilon P_{Y_F} L_F P_{Y_F}^{\frac{\lambda}{1-\lambda}}$$

and

$$P_{Y_F} = [\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda} \tag{101}$$

We can also get 101 by calculating 93, which is much more complicated, and omitted here.

5.4 Level Effect When Trade Opens

5.4.1 Externality

Recall that we have assumed that

$$(\frac{Z_{H1}^{\frac{\delta(1-\lambda)}{\lambda}}}{P_{G_{H1}}})^{\frac{\lambda}{1-\lambda}} \geq (\frac{Z_{F1}^{\frac{\delta(1-\lambda)}{\lambda}}}{P_{G_{F1}}})^{\frac{\lambda}{1-\lambda}} \quad \textbf{and} \quad (\frac{Z_{H2}^{\frac{\delta(1-\lambda)}{\lambda}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} \leq (\frac{Z_{F2}^{\frac{\delta(1-\lambda)}{\lambda}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}}$$

which implies

$$\frac{P_{G_{H1}}}{Z_{H1}^{\frac{\delta(1-\lambda)}{\lambda}}} \leq \frac{P_{G_{F1}}}{Z_{F1}^{\frac{\delta(1-\lambda)}{\lambda}}} \quad \textbf{and} \quad \frac{P_{G_{H2}}}{Z_{H2}^{\frac{\delta(1-\lambda)}{\lambda}}} \geq \frac{P_{G_{F2}}}{Z_{F2}^{\frac{\delta(1-\lambda)}{\lambda}}}$$

Home produces domestic intermediate good 1 and imports intermediate good 2. From section (5.1), if the Home X_2 industry uses the foreign intermediate good 2, G_{F2}, then there is a spillover to the

first industry of $\widehat{Z_{H2}} = Z_{F2}$. If $Z_{F2} > Z_{H2}$, where Z_{H2} is the autarky quality level in intermediate good 2, then importing the foreign intermediate good 2 has a positive externality from industry 2 to industry 1. If instead $Z_{F2} < Z_{H2}$, importing the foreign intermediate good in industry 2 will have a negative externality on industry 1.

Rewrite 52, Home's final good production function under autarky, as

$$\begin{aligned} Y_H^{Autarky} &= \kappa^{'} Z_{H1}^{(\delta+\gamma)\epsilon+[1-(\delta+\gamma)](1-\epsilon)} Z_{H2}^{[1-(\delta+\gamma)]\epsilon+(\delta+\gamma)(1-\epsilon)} P_{G_{H1}}^{\frac{-\lambda\epsilon}{(1-\lambda)}} P_{G_{H2}}^{\frac{-\lambda(1-\epsilon)}{(1-\lambda)}} (\epsilon L_H)^{\epsilon}[(1-\epsilon)L_H] \\ &= \kappa^{'}[(\frac{Z_{H1}^{\delta+\gamma}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}}) Z_{H2}^{1-(\delta+\gamma)} \epsilon L_H]^{\epsilon}[(\frac{Z_{H2}^{\delta+\gamma}}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}) Z_{H1}^{1-(\delta+\gamma)}(1-\epsilon)L_H]^{1-\epsilon} \end{aligned} \tag{102}$$

where

$$\kappa^{'} = \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}} \epsilon^{\frac{\lambda\epsilon}{1-\lambda}}$$

From the derivation of 52, we see that

$$[(\frac{Z_{H1}^{\delta+\gamma}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}}) Z_{H2}^{1-(\delta+\gamma)} \epsilon L_H]^{\epsilon}$$

is the contribution from X_{H1} to Y_H, in which Z_{H2} is the spillover from X_{H2} and ϵL_H is the total labor employed in industry 1. Similarly,

$$[(\frac{Z_{H2}^{\delta+\gamma}}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}) Z_{H1}^{1-(\delta+\gamma)}(1-\epsilon)L_H]^{1-\epsilon}$$

is the contribution from $X_{H2.}$ to Y_H, in which Z_{H1} is the spillover from X_{H1}, and $(1-\epsilon)L_H$ is the total labor employed in industry 2.

After trade opens, Home continues producing the domestic intermediate good 1 but imports intermediate good 2. Home's final good production function becomes

$$Y_H^{Trade} = \kappa^{'}[(\frac{Z_{H1}^{\delta+\gamma}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}}) Z_{F2}^{1-(\delta+\gamma)} (\epsilon L_H)]^{\epsilon}[(\frac{Z_{F2}^{\delta+\gamma}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}}) Z_{H1}^{1-(\delta+\gamma)}(1-\epsilon)L_H]^{1-\epsilon} \tag{103}$$

Comparing 103 with 102, we see that imports affect final output through two channels. The first channel is the quality-adjusted price:

$$\frac{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}{Z_{H2}^{\delta+\gamma}} = [\frac{P_{G_{H2}}}{Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}]^{\frac{\lambda}{1-\lambda}}$$

Imports have a lower quality-adjusted price than the corresponding domestically produced goods, so the quality per price is higher:

$$\frac{Z_{F2}^{\delta+\gamma}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}} > \frac{Z_{H2}^{\delta+\gamma}}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}} \tag{104}$$

This channel has a positive effect on output. The other channel is the spillover (externality) to industry 1, $Z_{F2}^{1-(\delta+\gamma)}$. If $Z_{F2} > Z_{H2}$, then the externality is positive and increases final output through the first term in 103. If $Z_{F2} < Z_{H2}$, the externality is negative and reduces final output. When the processed-good firms in industry 2 chooses between foreign and domestic intermediate goods, they do not take the externality into account. They only care which goods give them a higher marginal product to labor, which is a lower quality-adjusted price. It therefore is possible for them to choose to import a good with worse quality than the domestic good, i.e. $Z_{F2} < Z_{H2}$, if the price of that good is so low that 104 holds. They import the good and benefit from a higher marginal product of the labor that uses that good, but they do not take into account that the lower Z_{F2} enters industry 1 as a negative externality. If the externality is strong enough, total final output can decrease because of this negative externality.

5.4.2 Parameter Conditions

We derive the necessary and sufficient condition for the level of output to fall when trade opens. We then use that result to derive a simple sufficient condition for output to increase when trade opens.

For the level of output to fall when trade opens, we must have

$$Y_H^{Trade} < Y_H^{Autarky}$$

We can see from 102 and 103 that

$$\begin{aligned} Y_H^{Trade} &< Y_H^{Autarky} \Leftrightarrow \frac{(Z_{F2}^{\delta+\gamma})^{1-\epsilon}(Z_{F2}^{1-(\delta+\gamma)})^{\epsilon}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}} < \frac{(Z_{H2}^{\delta+\gamma})^{1-\epsilon}(Z_{H2}^{1-(\delta+\gamma)})^{\epsilon}}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}} \\ &\Leftrightarrow \left[\frac{\frac{(Z_{F2}^{\delta+\gamma})}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}}}{\frac{(Z_{H2}^{\delta+\gamma})}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}}\right]^{\frac{1-\epsilon}{\epsilon[1-(\delta+\gamma)]}} \frac{Z_{F2}}{Z_{H2}} < 1 \qquad (105) \\ &\Leftrightarrow \frac{Z_{H2}}{Z_{F2}} > \left(\frac{P_{G_{H2}}}{P_{G_{F2}}}\right)^{\frac{\lambda(1-\epsilon)}{(1-\lambda)[\epsilon+(\delta+\gamma)-2\epsilon(\delta+\gamma)]}} \qquad (106) \end{aligned}$$

Equation 105 is the necessary and sufficient condition for Y to fall when trade opens. Equation 106 is an alternative form that is useful below. Recall that we are assuming that Home specializes in type-1 goods and Foreign specializes in type-2 goods, a trade pattern that requires that inequality 84 be satisfied. That inequality implies that the first term on the left side of 105 is greater than 1:

$$\left[\frac{\frac{(Z_{F2}^{\delta+\gamma})}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}}}{\frac{(Z_{H2}^{\delta+\gamma})}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}}\right]^{\frac{1-\epsilon}{\epsilon[1-(\delta+\gamma)]}} > 1$$

That result together with 105 then implies that

$$\begin{aligned} \frac{Z_{F2}}{Z_{H2}} &< 1 \qquad (107) \\ \Leftrightarrow Z_{F2} &< Z_{H2} \end{aligned}$$

Therefore a sufficient condition for Y to rise when trade opens is

$$Z_{F2} > Z_{H2}$$

The second inequality in 84 can be written

$$\begin{aligned} \frac{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}} &> \frac{Z_{H2}^{\delta+\gamma}}{Z_{F2}^{\delta+\gamma}} \\ \Leftrightarrow \left(\frac{P_{G_{H2}}}{P_{G_{F2}}}\right)^{\frac{\lambda}{(1-\lambda)(\delta+\gamma)}} &> \left(\frac{Z_{H2}}{Z_{F2}}\right) \qquad (108) \end{aligned}$$

Combining 108 and 106 gives the interval within which the quality ratioZ_{H2}/Z_{F2} must fall for trade to have the pattern we have assumed and to reduce current output when trade opens:

$$\left(\frac{P_{G_{H2}}}{P_{G_{F2}}}\right)^{\frac{\lambda}{(1-\lambda)(\delta+\gamma)}} > \frac{Z_{H2}}{Z_{F2}} > \left(\frac{P_{G_{H2}}}{P_{G_{F2}}}\right)^{\frac{\lambda(1-\epsilon)}{(1-\lambda)[\epsilon+(\delta+\gamma)-2\epsilon(\delta+\gamma)]}} \qquad (109)$$

5.5 The World Balanced Growth Rate

We have an integrated economy with nine unknowns and nine equations as in section (4.1.1) plus the trade balance condition, which is solved in section (5.3) to get P_{Y_F}. The whole dynamic system reduces to 26, 33, 35, and 36,, with all $\alpha_1\theta_1$ replaced by $\alpha_{H1}\theta_{H1}$ and all $\alpha_2\theta_2$ replaced by $\alpha_{F2}\theta_{F2}$. The price of $G_1$is the Home price given by 22, $P_{G_{H1}} = A_{H1}/\lambda$. The price of G_2 is the Foreign price given by 30 converted to units of the numeraire, $P_{G_{F2}} = P_{Y_F}A_{F2}/\lambda$. As in the closed economy, prices do not affect growth rate, so the changes in parameters in the prices change nothing of substance. The Euler equation for both countries is the same in the closed economy. Following exactly the same steps as Section (4) to get the balanced growth rate:

$$\begin{aligned} g^\star_H &= \frac{\delta}{1-\delta}\sqrt{\alpha_{H1}\theta_{H1}\alpha_{F2}\theta_{F2}} - \frac{1}{1-\delta}\rho \\ &= g^\star_F \end{aligned} \tag{110}$$

where $(Z_{H1}/Z_{F2})^\star$ is given by 67 with $\alpha_1\theta_1 = \alpha_{H1}\theta_{H1}$ and $\alpha_2\theta_2 = \alpha_{F2}\theta_{F2}$.

5.6 Market Size After Trade

Following the same steps as in section (4.1), we get

$$L_{H1} = \epsilon L_H ...; \quad L_{H2} = (1-\epsilon)L_H$$
$$L_{F1} = \epsilon L_F \quad ; \quad L_{F2} = (1-\epsilon)L_F$$

so

$$\begin{aligned} L_{H1} + P_{Y_F}^{\frac{1}{1-\lambda}} L_{F1} &= \epsilon L_H + \{[\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda}\}^{\frac{1}{1-\lambda}}\epsilon L_F \\ &= \epsilon L_H + (1-\epsilon)L_H \\ &= L_H \end{aligned}$$

which is the total market size that the intermediate good firms in Home's industry 1 face. Trade increases the total size for Home's industry 1 from ϵL_H to the whole Home population. Similarly,

$$\begin{aligned} P_{Y_F}^{\frac{-1}{1-\lambda}} L_{H2} + L_{F2} &= \{[\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda}\}^{\frac{-1}{1-\lambda}}(1-\epsilon)L_H + (1-\epsilon)L_F \\ &= \epsilon L_F + (1-\epsilon)L_F \\ &= L_F \end{aligned}$$

which is the total market size that Foreign's intermediate good firms in industry 2 face., equal to the whole foreign population.

Continuing to follow the steps in section (4.1), we use 110, the Euler equation 54, and 128 to get the market size for each firm in Home:

$$l^\star_{H1} = \frac{L_H}{N^\star_{H1}} = \frac{\frac{\delta}{1-\delta}\sqrt{\alpha_{H1}\theta_{H1}\alpha_{F2}\theta_{F2}} - \frac{\delta}{1-\delta}\rho}{\alpha_{H1}\delta A_{H1}\frac{1-\lambda}{\lambda}[\frac{\lambda^2\epsilon}{A_{H1}}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}}(\frac{A_{H1}}{A_{F2}})^{\frac{\lambda(1-\epsilon)}{1-\lambda}}(\sqrt{\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}}})^{\Gamma-1}} \tag{111}$$

Recall that autarky market size for each firm is 73:

$$l^\star_{H1} = \frac{\epsilon L_H}{N^\star_{H1}} = \frac{\frac{\delta}{1-\delta}\sqrt{\alpha_{H1}\theta_{H1}\alpha_{H2}\theta_{H2}} - \frac{\delta}{1-\delta}\rho}{\alpha_{H1}\delta A_{H1}\frac{1-\lambda}{\lambda}[\frac{\lambda^2\epsilon}{A_{H1}}]^{\frac{1}{1-\lambda}}(\frac{\epsilon}{1-\epsilon})^{\frac{\epsilon-1}{1-\lambda}}(\frac{A_{H1}}{A_{H2}})^{\frac{\lambda(1-\epsilon)}{1-\lambda}}(\sqrt{\frac{\alpha_{H2}\theta_{H2}}{\alpha_{H1}\theta_{H1}}})^{\Gamma-1}}$$

Trade has led to a replacement of the term $\alpha_{H2}\theta_{H2}$ by $\alpha_{F2}\theta_{F2}$. It is straightforward to show that

$$\frac{\partial l^\star_{H1}}{\partial(\alpha_2\theta_2)} > 0$$

but in general we do not know if $\alpha_{H2}\theta_{H2} \gtrless \alpha_{F2}\theta_{F2}$, and so we do not know if trade raises or lowers firm size. However, we do see from 111 that there is once again no scale effect, exactly as in autarky, as explained in section (4.1).

5.7 Transition Dynamics

Similar to the closed economy, we have

$$r_{H1} = f_{H2}(\frac{Z_{H1}}{Z_{F2}}, \frac{L_H}{N_{H1}})$$

$$r_{F2} = f_{F2}(\frac{Z_{H1}}{Z_{F2}}, \frac{L_F}{N_{F2}})$$

$$g_{H1} = \frac{f_{H2}(\frac{Z_{H1}}{Z_{F2}}, \frac{L_H}{N_{H1}})}{\delta} - \alpha_{H1}\theta_{H1}\frac{Z_{H1}}{Z_{F2}}$$

$$g_{F2} = \frac{f_{F2}(\frac{Z_{H1}}{Z_{F2}}, \frac{L_F}{N_{F2}})}{\delta} - \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}$$

The no-arbitrage condition does not hold across countries because there is no international investment. However, from 99 and 100, we have

$$\frac{\dot{Y_H}}{Y_H} = \frac{\dot{Y_F}}{Y_F} = \Gamma\frac{\dot{Z_{H1}}}{Z_{H1}} + (1-\Gamma)\frac{\dot{Z_{F2}}}{Z_{F2}}$$

We assume identical preference functions in the two countries, so by the Euler equation 54,

$$\begin{aligned} \frac{\dot{c_H}}{c_H} &= \frac{\dot{c_F}}{c_F} = r_{H1} - \rho = r_{F2} - \rho \\ &\Leftrightarrow \quad r_{H1} = r_{F2} \end{aligned}$$

The the rate of return is equalized across the two countries, even without international investment. The transitional dynamics are similar to those in the closed economy:

$$\begin{aligned} \frac{\dot{(Z_{H1}/Z_{F2})}}{(Z_{H1}/Z_{F2})} &= g_{H1} - g_{F2} \\ &= \frac{r_{H1}}{\delta} - \alpha_{H1}\theta_{H1}\frac{Z_{H1}}{Z_{F2}} - (\frac{r_{F2}}{\delta} - \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}) \\ &= -\alpha_{H1}\theta_{H1}\frac{Z_{H1}}{Z_{F2}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} \end{aligned}$$

and

$$\dot{(Z_{H1}/Z_{F2})} = -\alpha_{H1}\theta_{H1}(\frac{Z_{H1}}{Z_{F2}})^2 + \alpha_{F2}\theta_{F2}$$

As in the closed economy, the positive steady state of the quality ratio Z_{H1}/Z_{F2} is stable.

6 Open Economy under Incomplete Specialization

6.1 Define Comparative Advantage

The condition for our maintained trade pattern is the double inequality 85. Complete specialization occurs when both inequalities are strict. In that case, the solution for P_{Y_F} is given by 101, repeated here:

$$P_{Y_F} = [\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda}$$

The price P_{Y_F} must fall inside the upper and lower bounds of the double inequality 85 because otherwise both countries would try to export the same good and trade balance would be violated. However, there is no reason that the expression on the right side of 101 need be inside the trade pattern interval. When it is outside that interval, then P_{Y_F} cannot be equal to it. Instead, P_{Y_F} will equal whichever of the two trade pattern bounds is closer to $[(1-\epsilon)L_H/\epsilon L_F]^{1-\lambda}$. One of the two countries will continue to specialize completely but the other will not specialize and instead will continue to produce both types of intermediate goods. For concreteness and with no loss of generality, we assume that $[(1-\epsilon)L_H/\epsilon L_F]^{1-\lambda}$ is large enough that

$$[\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda} \geq \frac{A_{H2}/Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{A_{F2}/Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} = P_{Y_F} > \frac{A_{H1}/Z_{H1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{A_{F1}/Z_{F1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \tag{112}$$

All the results we derive in this section hold in mirror image if $[(1-\epsilon)L_H/\epsilon L_F]^{1-\lambda}$ small enough to be below the smaller bound of the trade pattern interval. Under this assumption, the quality-adjusted price of intermediate good 1 is lower for Home's product. The quality-adjusted price of intermediate good 2 is the same for both Home and Foreign products. Home produces both types of intermediates, and Foreign specializes in producing intermediate good 2. Home exports intermediate good 1 and imports intermediate good 2. Intuitively, Foreign is not "big" enough to fulfill Home's demand for good 2, so home country produces good 2 as well as good 1.

6.2 Model Setup

6.2.1 Home Country: Final Good and Processed Good

Home's final good and processed good production functions are

$$Y_H = X_{H1}^{\epsilon} X_{H2}^{1-\epsilon}$$

$$X_{H1} = \int_0^{N_{H1}} (G_{H1} - G_{H1}^E)^{\lambda} [Z_{H1}^{(\delta+\gamma)} (\widetilde{Z_{H2}})^{1-(\delta+\gamma)}]^{1-\lambda} dj$$

$$X_{H2} = \int_0^{N_{H2}} G_{H2}^{\lambda} (Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2H})^{1-\lambda} dj + \int_0^{N_{F2}} (G_{H2}^I)^{\lambda} (Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2F})^{1-\lambda} dj$$

Home exports G_{H1}^E units of good 1. Home uses both domestic and foreign goods in industry 2. Denote by $L_{H2H} = \int_0^{N_{H2}} l_{H2H} dj$ and $L_{H2F} = \int_0^{N_{F2}} l_{H2F} dj$ the amount of labor in Home using domestic and foreign intermediate good 2, respectively. Following similar steps as in section (1), we get the inverse demands for X_{H1} and X_{H2}:

$$P_{X_{H1}} = \epsilon X_{H1}^{\epsilon-1} X_{H2}^{1-\epsilon} \tag{113}$$

$$P_{X_{H2}} = (1-\epsilon) X_{H1}^{\epsilon} X_{H2}^{-\epsilon} \tag{114}$$

and also the demand functions for intermediate goods and labor:

$$G_{H1j} - G_{H1j}^E = (\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}})^{\frac{1}{1-\lambda}} Z_{H1j}^{\delta} Z_{H1}^{\gamma} (\widetilde{Z_{H2}})^{1-(\delta+\gamma)} \tag{115}$$

$$G_{H2j} = (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{1}{1-\lambda}} Z_{H2j}^{\delta} Z_{H2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} l_{H1H} \tag{116}$$

$$G_{H2j}^I = (\frac{\lambda P_{X_{H2}}}{P_{G_{F2}}})^{\frac{1}{1-\lambda}} Z_{F2j}^{\delta} Z_{F2}^{\gamma} Z_{H1}^{1-(\delta+\gamma)} l_{H1F} \tag{117}$$

$$l_{H1} = (P_{X_{H1}} \frac{1-\lambda}{W_H})^{\frac{1}{\lambda}} (G_{H1} - G_{H1}^E)[Z_{H1}^{(\delta+\gamma)} (\widetilde{Z_{H2}})^{1-(\delta+\gamma)}] \tag{118}$$

$$l_{H2H} = (P_{X_{H2}} \frac{1-\lambda}{W_H})^{\frac{1}{\lambda}} G_{H2} [Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}]^{\frac{1-\lambda}{\lambda}} \tag{119}$$

$$l_{H2F} = (P_{X_{H2}} \frac{1-\lambda}{W_H})^{\frac{1}{\lambda}} G^I_{F2} [Z^{(\delta+\gamma)}_{F2} Z^{1-(\delta+\gamma)}_{H1}]^{\frac{1-\lambda}{\lambda}} \tag{120}$$

The marginal product of Home labor in the X_1 industry is

$$\begin{aligned} MPl_{H1} &= P_{X_{H1}} \frac{\partial X_{H1}}{\partial l_{H1}} \\ &= P_{X_{H1}}(1-\lambda)(G_{H1} - G^E_{H1})^{\lambda} [Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1}]^{-\lambda} Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} \end{aligned}$$

Using the demand function 115, we get

$$\begin{aligned} MPl_{H1} &= P_{X_{H1}}(1-\lambda)[(\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}})^{\frac{1}{1-\lambda}} Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1}]^{\lambda} \\ &\quad \bullet [Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1}]^{-\lambda} Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} \\ &= P_{X_{H1}}(1-\lambda)(\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}})^{\frac{\lambda}{1-\lambda}} Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} \end{aligned}$$

Similarly, we get

$$MPl_{H2H} = P_{X_{H2}}(1-\lambda)(\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} Z^{(\delta+\gamma)}_{H2} Z^{1-(\delta+\gamma)}_{H1}$$

and

$$MPl_{H2F} = P_{X_{H2}}(1-\lambda)(\frac{\lambda P_{X_{H2}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}} Z^{(\delta+\gamma)}_{F2} Z^{1-(\delta+\gamma)}_{H1}$$

All marginal products of labor must be equal in Home: $MPl_{H1} = MPl_{H2H} = MPl_{H2F}$. The latter equality, $MPl_{H2H} = MPl_{H2F}$, is equivalent to our trade pattern condition:

$$\begin{aligned} P_{H2}/Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} &= P_{F2}/Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} \\ \iff \frac{A_{H2}/Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{A_{F2}/Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} &= P_{Y_F} \end{aligned}$$

where $P_{H2} = A_{H2}/\lambda$ and $P_{F2} = P_{Y_F} A_{F2}/\lambda$. Domestic and foreign goods yield the same marginal product for labor, so Home chooses both goods. Equality of the three marginal products also implies

$$(\frac{P_{X_{H1}}}{P_{X_{H2}}})^{\frac{1}{1-\lambda}} = (\frac{P_{G_{H1}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z^{(\delta+\gamma)}_{H2} Z^{1-(\delta+\gamma)}_{H1}}{Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)}} = (\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z^{(\delta+\gamma)}_{F2} Z^{1-(\delta+\gamma)}_{H1}}{Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)}} \tag{121}$$

Now we use $P_{G_{ij}}$ and Z_{ij} to express $P_{X_{H1}}$ and $P_{X_{H2}}$. Divide 113 by 114 and rearrange terms to get X_{H1}/X_{H2} as a function of $P_{X_{H1}}/P_{X_{H2}}$:

$$\begin{aligned} \frac{P_{X_{H1}}}{P_{X_{H2}}} &= \frac{\epsilon}{1-\epsilon}(\frac{X_{H1}}{X_{H2}})^{-1} \\ \Rightarrow \frac{X_{H1}}{X_{H2}} &= \frac{\epsilon}{1-\epsilon}(\frac{P_{X_{H1}}}{P_{X_{H2}}})^{-1} \end{aligned}$$

Next substitute 121 into the right side of this last equation:

$$\begin{aligned} \frac{X_{H1}}{X_{H2}} &= \frac{\epsilon}{1-\epsilon}(\frac{P_{X_{H1}}}{P_{X_{H2}}})^{-1} \\ &= \frac{\epsilon}{1-\epsilon}[(\frac{P_{G_{H1}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z^{(\delta+\gamma)}_{H2} Z^{1-(\delta+\gamma)}_{H1}}{Z^{(\delta+\gamma)}_{H1} (\widehat{Z_{H2}})^{1-(\delta+\gamma)}}]^{-(1-\lambda)} \end{aligned}$$

Substitute this expression back into the inverse demand functions 113 and 114 to get $P_{X_{H1}}$ and $P_{X_{H2}}$:

$$P_{X_{H1}} = \epsilon\{\frac{\epsilon}{1-\epsilon}[(\frac{P_{G_{H1}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} \widehat{(Z_{H2})^{1-(\delta+\gamma)}}}]^{-(1-\lambda)}\}^{\epsilon-1} \tag{122}$$

By the trade balance condition

$$P_{H2}/Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} = P_{F2}/Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}$$

we can write $P_{X_{H1}}$ as

$$P_{X_{H1}} = \epsilon\{\frac{\epsilon}{1-\epsilon}[(\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} \widehat{(Z_{H2})^{1-(\delta+\gamma)}}}]^{-(1-\lambda)}\}^{\epsilon-1}$$

Similarly,

$$\begin{aligned} P_{X_{H2}} &= (1-\epsilon)\{\frac{\epsilon}{1-\epsilon}[(\frac{P_{G_{H1}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} \widehat{(Z_{H2})^{1-(\delta+\gamma)}}}]^{-(1-\lambda)}\}^{\epsilon} \\ &= (1-\epsilon)\{\frac{\epsilon}{1-\epsilon}[(\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} \widehat{(Z_{H2})^{1-(\delta+\gamma)}}}]^{-(1-\lambda)}\}^{\epsilon} \end{aligned} \tag{123}$$

Home producers in processed good industry 2 use both domestic and foreign intermediate goods

$$X_{H2} = \int_0^{N_{H2}} G_{H2}^{\lambda} (Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2H})^{1-\lambda} dj + \int_0^{N_{F2}} (G_{F2}^{I})^{\delta} (Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2F})^{1-\lambda} dj$$

Denote

$$\begin{aligned} X_{H2H} &= \int_0^{N_{H2}} G_{H2}^{\lambda} (Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2H})^{1-\lambda} dj \\ X_{H2F} &= \int_0^{N_{F2}} (G_{F2}^{I})^{\lambda} (Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2F})^{1-\lambda} dj \end{aligned}$$

Plug the demand functions 116 and 117 inside these expressions and take note of symmetry to get

$$\begin{aligned} X_{H2H} &= N_{H2} (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} (Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2H})^{\lambda} (Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{H2H})^{1-\lambda} \\ &= (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} L_{H2H} \end{aligned}$$

where $L_{H2H} = N_{H2} l_{H2H}$ and

$$X_{H2F} = (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} L_{H2F}$$

where $L_{H2F} = N_{F2} l_{H2F}$. Thus, $X_{H2H}/X_{H2F} = L_{H2H}/L_{H2F}$, given the trade pattern condition $Z_{H2}^{(\delta+\gamma)}/P_{G_{H2}}^{\frac{\lambda}{1-\lambda}} = Z_{F2}^{(\delta+\gamma)}/P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}$. So $X_{H2H} = (L_{H2H}/L_{H2})X_{H2}$ and $X_{H2F} = (L_{H2F}/L_{H2})X_{H2}$, where $L_{H2H} + L_{H2F} = L_{H2}$. Producers using X_{H2H} and X_{H2F} both devote the proportion λ of their product to compensation for labor, L_{H2H} and L_{H2F}, so the entire X_{H2} industry devotes the proportion λ of output to compensate the labor total $L_{H2} = L_{H2H} + L_{H2F}$.

By resource allocation as in section (1), we get

$$\frac{L_{H1}}{L_{H2H} + L_{H2F}} = \frac{\epsilon}{1-\epsilon} \tag{124}$$

We can also write

$$\begin{aligned} X_{H2} &= (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} Z_{H2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} (L_{H2H} + L_{H2F}) \\ &= (\frac{\lambda P_{X_{H2}}}{P_{G_{H2}}})^{\frac{\lambda}{1-\lambda}} Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} (L_{H2H} + L_{H2F}) \end{aligned}$$

6.2.2 Foreign Country: Final Good and Processed Good

Foreign's production functions are

$$Y_F = X_{F1}^{\epsilon} X_{F2}^{1-\epsilon}$$

$$X_{F1} = \int_0^{N_{H1}} (G_{F1}^I)^{\lambda} [Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} l_{F1}]^{1-\lambda} \, dj$$

$$X_{F2} = \int_0^{N_{F2}} (G_{F_2} - G_{F2}^E)^{\lambda} [Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)} l_{F2}]^{1-\lambda} \, dj$$

The price of Y_F is P_{Y_F}, Foreign imports $G_{F1}^I = G_{H1}^E$ from Home, and exports $G_{F2}^E = G_{H2}^I$ to Home. The inverse demand functions for X_{F1} and X_{F2} are

$$P_{X_{F1}} = P_{Y_F} \epsilon X_{F1}^{\epsilon-1} X_{F2}^{1-\epsilon}$$

$$P_{X_{F2}} = P_{Y_F} (1-\epsilon) X_{F1}^{\epsilon} X_{F2}^{-\epsilon}$$

The demands for intermediate goods are

$$G_{H1}^E = G_{F1}^I = [\frac{\lambda P_{X_{F1}}}{P_{G_{H1}}}]^{\frac{1}{1-\lambda}} Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} l_{F1}$$

$$G_{F_2} - G_{F2}^E = [\frac{\lambda P_{X_{F2}}}{P_{G_{F2}}}]^{\frac{1}{1-\lambda}} Z_{F2}^{(\delta+\gamma)} (Z_{H1})^{1-(\delta+\gamma)} l_{H2}$$

From $MPl_{F1} = MPl_{F2}$, we get

$$(\frac{P_{X_{F1}}}{P_{X_{F2}}})^{\frac{1}{1-\lambda}} = (\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)}}$$

Following the same steps as in the derivation of 122 and 123, we get

$$P_{X_{F1}} = \epsilon \{ \frac{\epsilon}{1-\epsilon} [(\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)}}]^{-(1-\lambda)} \}^{\epsilon-1} \tag{125}$$

$$P_{X_{F2}} = (1-\epsilon) \{ \frac{\epsilon}{1-\epsilon} [(\frac{P_{G_{H1}}}{P_{G_{F2}}})^{\frac{\lambda}{1-\lambda}} \frac{Z_{F2}^{(\delta+\gamma)} Z_{H1}^{1-(\delta+\gamma)}}{Z_{H1}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)}}]^{-(1-\lambda)} \}^{\epsilon} \tag{126}$$

As before, resource allocation implies

$$\frac{N_{H1} l_{F1}}{N_{F2} l_{F2}} = \frac{L_{F1}}{L_{F2}} = \frac{\epsilon}{1-\epsilon}$$

6.2.3 Home Country: Intermediate Good 1

The total demand facing Home's intermediate good firms in industry 1 is

$$\begin{aligned} G_{H1} &= (G_{H1} - G_{H1}^E) + G_{F1}^I \\ &= (\frac{\lambda P_{X_{H1}}}{P_{G_{H1}}})^{\frac{1}{1-\lambda}} Z_{H1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1} + [\frac{\lambda P_{X_{F1}}}{P_{G_{H1}}}]^{\frac{1}{1-\lambda}} Z_{H1j}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} l_{F1} \\ &= P_{G_{H1}}^{-\frac{1}{1-\lambda}} [(\lambda P_{X_{H1}})^{\frac{1}{1-\lambda}} Z_{H1}^{(\delta+\gamma)} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1} + (\lambda P_{X_{F1}})^{\frac{1}{1-\lambda}} Z_{H1j}^{(\delta+\gamma)} Z_{F2}^{1-(\delta+\gamma)} l_{F1}] \end{aligned} \tag{127}$$

The cash flow to those firms is

$$F_{H1} = G_{H1}(P_{G_{H1}} - A_{H1}) - \theta_{H1} \frac{Z_{H1}^2}{\widehat{Z_{H2}}}$$

The firm maximizes its current value Hamiltonian

$$CVH_{H1} = G_{H1}(P_{G_{H1}} - A_{H1}) - \theta_{H1} \frac{Z_{H1}^2}{\widehat{Z_{H2}}} - R_{H1} + q_{H1}(\alpha_{H1} R_{H1})$$

s.t. the demand function for G_{H1}, which is 127, and the quality-improvement production function $\dot{Z_{H1}} = \alpha_{H1} R_{H1}$. Following similar steps as in section (2), we get the necessary conditions

$$\dot{Z_{H1}} = \alpha_{H1} R_{H1}$$

$$\frac{\partial CVH_{H1}}{\partial P_{G_{H1}}} = 0 \Rightarrow P_{G_{H1}} = \frac{A_{H1}}{\lambda}$$

$$\frac{\partial CVH_{H1}}{\partial R_{H1}} = -1 + q_{H1}\alpha_{H1} \Rightarrow \begin{cases} q_{H1} < \frac{1}{\alpha_{H1}}, & R_{H1} = 0 \\ q_{H1} = \frac{1}{\alpha_{H1}} & R_{H1} > 0 \quad (\text{we focus on this case}) \\ q_{H1} > \frac{1}{\alpha_{H1}} & R_{H1} = \infty \end{cases}$$

$$Z_{H1,t=0} \quad \text{given}$$

$$lim_{t\to\infty} e^{-\int_t^\tau r(s)ds} q_{H1}(t) Z_{H1}(t) = 0$$

$$\begin{aligned} \dot{q_{1j}} &= -r_{1j} q_{1j} + \frac{\partial CVH_{1j}}{\partial Z_{1j}} \\ \Rightarrow r_{H1} &= \frac{\partial F_{H1}}{\partial Z_{H1}} / q_{H1} + \frac{\dot{q_{H1}}}{q_{H1}} \end{aligned}$$

It is convenient to write r_{H1} as

$$r_{H1} = \delta f_{H1}(\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, N_{H1}) \tag{128}$$

where

$f_{H1} = \alpha_{H1} \frac{\lambda}{1-\lambda} A_{H1} Z_{H1}^{(\delta+\gamma)-1} \{(\frac{\lambda P_{X_{H1}}}{A_{H1}/\lambda})^{\frac{1}{1-\lambda}} (\widehat{Z_{H2}})^{1-(\delta+\gamma)} l_{H1} + (\frac{\lambda P_{X_{F1}}}{A_{H1}/\lambda})^{\frac{1}{1-\lambda}} (Z_{F2})^{1-(\delta+\gamma)} l_{F1}\}$
$l_{H1} = \frac{\epsilon L_H}{N_{H1}}$
$l_{F1} = \frac{\epsilon L_F}{N_{H1}}$
$P_{X_{H1}}$ is a function of unit costs, $\frac{Z_{H1}}{Z_{H2}}$, and $\frac{Z_{H1}}{Z_{F1}}$, and is given by 122
$P_{X_{F1}}$ is a function of unit costs and $\frac{Z_{H1}}{Z_{F2}}$, and is given by 125

Zero entry cost ensures that the zero profit condition holds, so

$$g_{H1} = \frac{\alpha_{H1} R_{H1}}{Z_{H1}} = \frac{\alpha_{H1} F_{H1}}{Z_{H1}} = f_{H1}(\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, N_{H1}) - \alpha_{H1}\theta_{H1} \frac{Z_{H1}}{\widehat{Z_{H2}}} \tag{129}$$

6.2.4 Home Country: Intermediate Good 2

Firms in Home's industry 2 produce goods that are used only by Home country. Firms' cash flow is

$$F_{H2} = G_{H2}(P_{G_{H2}} - A_{H2}) - \theta_{H2}\frac{Z_{H2}^2}{Z_{H1}}$$

The firm's problem is to maximize

$$CVH_{H2} = G_{H2}(P_{G_{H2}} - A_{H2}) - \theta_{H2}\frac{Z_{H2}^2}{Z_{H1}} - R_{H2} + q_{H2}(\alpha_{H2}R_{H2})$$

s.t. the demand function for G_{H2}, which is 116, and the quality-improvement production function $\dot{Z_{H2}} = \alpha_{H2}R_{H2}$. Once again following similar steps as in section (2), we get the necessary conditions:

$$\dot{Z_{H2}} = \alpha_{H2}R_{H2}$$

$$\frac{\partial CVH_{H2}}{\partial P_{G_{H2}}} = 0 \Rightarrow P_{G_{H2}} = \frac{A_{H2}}{\lambda}$$

$$\frac{\partial CVH_{H2}}{\partial R_{H2}} = -1 + q_{H2}\alpha_{H2} \Rightarrow \begin{cases} q_{H2} < \frac{1}{\alpha_{H2}}, & R_{H2} = 0 \\ q_{H2} = \frac{1}{\alpha_{H2}} & R_{H2} > 0 \quad \text{(we focus on this case)} \\ q_{H2} > \frac{1}{\alpha_{H2}} & R_{H2} = \infty \end{cases}$$

$$Z_{H2,t=0} \quad \text{given}$$

$$lim_{t\to\infty} e^{-\int_t^\tau r(s)ds} q_{H2}(t)\, Z_{H2}(t) = 0$$

$$\begin{aligned} \dot{q_{H2j}} &= -r_{H2j}q_{H2j} + \frac{\partial CVH_{H2j}}{\partial Z_{H2j}} \\ \Rightarrow \quad r_{H2} &= \frac{\partial F_{H2}}{\partial Z_{H2}}/q_{H2} + \frac{\dot{q_{H2}}}{q_{H2}} \end{aligned}$$

As with intermediate good 1, we write r_{H2} in the simplified form

$$r_{H2} = \delta f_{H2}(\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, \frac{Z_{F2}}{Z_{H2}}, N_{H2}) \tag{130}$$

where

$f_{H2} = \alpha_{H2}\frac{\lambda}{1-\lambda}A_{H2}(\frac{\lambda^2 P_{X_{H2}}}{A_{H2}})^{\frac{1}{1-\lambda}}(\frac{Z_{H2}}{Z_{H1}})^{(\delta+\gamma)-1}l_{H2H}$
$l_{H2H} = \frac{L_{H2H}}{N_{H2}}$
$L_{H2H} = (1-\epsilon)L_H - L_{H2F}$
L_{H2F} is a function of $\frac{Z_{F2}}{Z_{H2}}$ by trade balance condition, 138
$P_{X_{H2}}$ is a function of $\frac{Z_{H1}}{Z_{F1}}$ and $\frac{Z_{H1}}{Z_{F2}}$ and is given by 123

Zero entry cost implies that the zero profit condition always is satisfied, so

$$\begin{aligned} \frac{\alpha_{H2}R_{H2}}{Z_{H2}} &= \frac{\alpha_{H2}F_{H2}}{Z_{H2}} \\ &= f_{H2}(\frac{Z_{H1}}{Z_{H2}}, \frac{Z_{H1}}{Z_{F2}}, \frac{Z_{F2}}{Z_{H2}}, N_{H2}) - \alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} \\ &= g_{H2} \end{aligned} \tag{131}$$

6.2.5 Foreign Country: Intermediate Good 2

Firms in Foreign industry 2 produce goods used by both Home and Foreign. The demand function is

$$G_{F2} = [\frac{\lambda}{P_{G_{F2}}}]^{\frac{1}{1-\lambda}} Z_{F2}^{\delta}(Z_{H1})^{1-\delta}(P_{X_{H2}}^{\frac{1}{1-\lambda}} l_{H2F} + P_{X_{H2}}^{\frac{1}{1-\lambda}} l_{F2}) \tag{132}$$

The cash flow is

$$F_{F2} = G_{F2}(P_{G_{F2}} - P_{Y_F} A_{F2}) - P_{Y_{F2}} \theta_{F2} \frac{Z_{F2}^2}{Z_{H1}}$$

The firm maximizes

$$CVH_{F2} = G_{F2}(P_{G_{F2}} - P_{Y_F} A_{F2}) - P_{Y_{F2}} \theta_{F2} \frac{Z_{F2}^2}{Z_{H1}} - P_{Y_F} R_{F2} + q_{F2}(\alpha_{F2} R_{F2})$$

s.t. the demand function 132; and the quality-improvement production function $\dot{Z}_{F2} = \alpha_{F2} R_{F2}$. The necessary conditions are

$$\dot{Z}_{F2} = \alpha_{F2} R_{F2}$$

$$\frac{\partial CVH_{F2}}{\partial P_{GF2}} = 0 \Longrightarrow P_{G_{F2}} = P_{Y_F} \frac{A_{F2}}{\lambda}$$

$$\frac{\partial CVH_{F2}}{\partial R_{F2}} = -P_{Y_F} + q_{F2}\alpha_{F2} \Rightarrow Interior\ solution: \ q_{2F} = \frac{P_{Y_2}}{\alpha_{F2}}$$

$$Z_{F2,t=0} is given$$

$$lim_{t\to\infty} e^{-\int_t^\tau r(s)ds} q_{F2j}(t)\, Z_{F2j}(t) = 0$$

$$\dot{q_{F2}} = -r_{F2} q_{F2} + \frac{\partial CVH_{F2}}{\partial Z_{F2}} \Rightarrow r_{F2} = \frac{\partial F_{F2}}{\partial Z_{F2}} / q_{F2} + \frac{\dot{q_{F2}}}{q_{F2}}$$

We write r_{F2} in the simplified form

$$r_{F2} = \delta f_{F2}(\frac{Z_{H1}}{Z_{F2}}, \frac{Z_{H2}}{Z_{F2}}, \frac{Z_{H1}}{Z_{F1}}, N_{F2}) \tag{133}$$

where

$f_{F2} = \alpha_{F2} A_{F2} \frac{1-\lambda}{\lambda} (\frac{\lambda^2}{P_{Y_F} A_{F2}})^{\frac{1}{1-\lambda}} (\frac{Z_{F2}}{Z_{H1}})^{(\delta+\gamma)-1} (P_{X_{F2}}^{\frac{1}{1-\lambda}} l_{F2} + P_{X_{H2}}^{\frac{1}{1-\lambda}} l_{H2F})$
$l_{F2} = \frac{(1-\epsilon)L_F}{N_{F2}}$
$l_{H2F} = \frac{L_{H2F}}{N_{F2}}$ where L_{H2F} is a function of $\frac{Z_{H2}}{Z_{F2}}$ from 138
$P_{X_{F2}}$ is a function of $\frac{Z_{H1}}{Z_{F2}}$ in 126
$P_{X_{H2}}$ is a function of $\frac{Z_{H1}}{Z_{H2}}$ and $\frac{Z_{H1}}{Z_{F2}}$ in 123
P_{Y_F} is a function of $\frac{Z_{H2}}{Z_{F2}}$ by the trade pattern condition 85

Free entry implies that the zero profit condition always holds, so

$$\begin{aligned} \frac{\alpha_{F2} R_{F2}}{Z_{F2}} &= \frac{\alpha_{F2} F_{F2}}{Z_{F2}} \\ &= f_{F2}(\frac{Z_{H1}}{Z_{F2}}, \frac{Z_{H2}}{Z_{F2}}, \frac{Z_{H1}}{Z_{F1}}, N_{F2}) - \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} \\ &= g_{F2} \end{aligned} \tag{134}$$

6.3 Trade Balance Condition

The trade balance condition that the value of Home's exports equals the value of its imports:

$$N_{H1}G^E_{H1}P_{G_{H1}} = N_{F2}G^I_{F2}P_{G_{F2}}$$

We can use the implications of Cobb-Douglas production and competitive markets to rewrite this condition in terms of fractions of product paid to factors of production. We have

$$\begin{aligned} N_{H1}G^E_{H1}P_{G_{H1}} &= P_{Y_F}\epsilon\lambda Y_F \\ N_{F2}G^I_{F2}P_{G_{F2}} &= Y_H(1-\epsilon)\lambda\frac{L_{H2F}}{L_{H2}} \end{aligned}$$

so the trade balance condition can be written as

$$P_{Y_F}\epsilon\lambda Y_F = Y_H(1-\epsilon)\lambda\frac{L_{H2F}}{L_{H2}} \tag{135}$$

We have

$$P_{Y_F} = \frac{A_{H2}/Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{A_{F2}/Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}$$

from 112. Following similar steps as in the derivation of 102, we get

$$Y_H^{Trade} = \kappa^{'}[(\frac{Z_{H1}^{(\delta+\gamma)}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}})(\widehat{Z_{H2}})^{1-(\delta+\gamma)}(\epsilon L_H)]^{\epsilon}[(\frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}})Z_{H1}^{1-(\delta+\gamma)}((1-\epsilon)L_H)]^{1-\epsilon} \tag{136}$$

and

$$Y_F^{Trade} = \kappa^{'}[(\frac{Z_{H1}^{(\delta+\gamma)}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}})Z_{F2}^{1-(\delta+\gamma)}P_{Y_F}^{\frac{\lambda}{1-\lambda}}(\epsilon L_F)]^{\epsilon}[(\frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}})Z_{H1}^{1-(\delta+\gamma)}P_{Y_F}^{\frac{\lambda}{1-\lambda}}((1-\epsilon)L_F)]^{1-\epsilon} \tag{137}$$

where

$$\kappa^{'} = \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}}$$

and

$$\frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}} = \frac{Z_{H2}^{(\delta+\gamma)}}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}$$

based on the trade pattern condition 112. Plug 136 , 137 and

$$P_{Y_F} = \frac{A_{H2}/Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{A_{F2}/Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}$$

into 135 to get

$$\begin{aligned} L_{H2F} &= \frac{\epsilon}{1-\epsilon}P_{Y_F}\frac{Y_F}{Y_H}L_{H2} \\ &= \frac{\epsilon}{1-\epsilon}P_{Y_F}P_{Y_F}^{\frac{\lambda}{1-\lambda}}[\frac{(\widehat{Z_{H2}})^{1-(\delta+\gamma)}}{Z_{F2}^{1-(\delta+\gamma)}}]^{\epsilon}\frac{L_F}{L_H}(1-\epsilon)L_H \\ &= \epsilon L_F P_{Y_F}^{\frac{1}{1-\lambda}}[\frac{(\widehat{Z_{H2}})^{1-(\delta+\gamma)}}{Z_{F2}^{1-(\delta+\gamma)}}]^{\epsilon} \\ &= \epsilon L_F(\frac{A_{H2}}{A_{F2}})^{\frac{1}{1-\lambda}}(\frac{Z_{F2}}{Z_{H2}})^{\frac{(\delta+\gamma)}{\lambda}}[\frac{(\widehat{Z_{H2}})^{1-(\delta+\gamma)}}{Z_{F2}^{1-(\delta+\gamma)}}]^{\epsilon} \end{aligned} \tag{138}$$

We see that L_{H2F} is a function of Z_{F2}/Z_{H2} because $\widehat{Z_{H2}} = Z_{H2}^{\eta}Z_{F2}^{1-\eta}$, and $L_{H2F} < L_{H2} = (1-\epsilon)L_H$ because Home allocates labor to both G_H goods and G_F goods.

6.4 Level Effect

Compare Home's level of final good output before and after trade at the moment that trade opens. Following similar steps as in section (5.4), we get the level of output before trade:

$$Y_H^{Autarky} = \kappa^{'}[(\frac{Z_{H1}^{(\delta+\gamma)}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}})(Z_{H2})^{1-(\delta+\gamma)}(\epsilon L_H)]^{\epsilon}[(\frac{Z_{H2}^{(\delta+\gamma)}}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}})Z_{H1}^{1-(\delta+\gamma)}((1-\epsilon)L_H)]^{1-\epsilon}$$

and equation 136 we get the level after trade:

$$Y_H^{Trade} = \kappa^{'}[(\frac{Z_{H1}^{(\delta+\gamma)}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}})(\widetilde{Z_{H2}})^{1-(\delta+\gamma)}(\epsilon L_H)]^{\epsilon}[(\frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}})Z_{H1}^{1-(\delta+\gamma)}((1-\epsilon)L_H)]^{1-\epsilon}$$

where $\kappa^{'} = \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}}$. The second term in the two expressions has the same value because, by assumption,

$$\frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}} = \frac{Z_{H2}^{(\delta+\gamma)}}{P_{G_{H2}}^{\frac{\lambda}{1-\lambda}}}$$

The only difference between the two income levels is the spillover term. Under autarky, that term depends only on the quality Z_{H2} of the domestic good, whereas with trade it depends on the qualities of both the domestic and foreign goods: $\widetilde{Z_{H2}} = Z_{H2}^{\eta}Z_{F2}^{1-\eta}$. Thus the necessary and sufficient condition for trade to increase level output at the moment of trade, i.e. $Y_H^{Trade} > Y_H^{Autarky}$ is $Z_{F2} > Z_{H2}$, namely, that the externality to processed goods industry 1 derived from imports of type-2 intermediate goods exceeds the externality derived from domestically produced intermediate goods.

Foreign's final good output before and after trade is

$$Y_F^{Autarky} = \kappa^{'}[(\frac{Z_{F1}^{(\delta+\gamma)}}{P_{G_{F1}}^{\frac{\lambda}{1-\lambda}}})Z_{F2}^{1-(\delta+\gamma)}P_{Y_F}^{\frac{\lambda}{1-\lambda}}(\epsilon L_F)]^{\epsilon}[(\frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}})Z_{H1}^{1-(\delta+\gamma)}P_{Y_F}^{\frac{\lambda}{1-\lambda}}((1-\epsilon)L_F)]^{1-\epsilon}$$

and

$$Y_F^{Trade} = \kappa^{'}[(\frac{Z_{H1}^{(\delta+\gamma)}}{P_{G_{H1}}^{\frac{\lambda}{1-\lambda}}})Z_{F2}^{1-(\delta+\gamma)}P_{Y_F}^{\frac{\lambda}{1-\lambda}}(\epsilon L_F)]^{\epsilon}[(\frac{Z_{F2}^{(\delta+\gamma)}}{P_{G_{F2}}^{\frac{\lambda}{1-\lambda}}})Z_{H1}^{1-(\delta+\gamma)}P_{Y_F}^{\frac{\lambda}{1-\lambda}}((1-\epsilon)L_F)]^{1-\epsilon}$$

Foreign is completely specialized, so the level effects at the moment when trade opens are exactly as in section (5.4).

6.5 Balanced Growth Rate Effect

Rewrite equations 136 and 137 as

$$Y_H^{Trade} = \tilde{\kappa}_H Z_{H1}^{\Gamma}Z_{H2}^{\eta[1-(\delta+\gamma)]\epsilon+(\delta+\gamma)(1-\epsilon)}Z_{F2}^{(1-\eta)[1-(\delta+\gamma)]\epsilon}$$

and

$$Y_F^{Trade} = \tilde{\kappa}_F Z_{H1}^{\Gamma}Z_{H2}^{-(\delta+\gamma)\epsilon}Z_{F2}^{(1-(\delta+\gamma))\epsilon+(\delta+\gamma)}$$

where the constants $\tilde{\kappa}_H$ and $\tilde{\kappa}_F$ are

$$\begin{aligned}
\tilde{\kappa}_H &= \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}}P_{G_{H1}}^{-\frac{\lambda\epsilon}{1-\lambda}}P_{G_{H2}}^{\frac{-\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon}L_H \\
&= \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}}(\frac{A_{H1}}{\lambda})^{-\frac{\lambda\epsilon}{1-\lambda}}(\frac{A_{H2}}{\lambda})^{\frac{-\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon}L_H \\
\tilde{\kappa}_F &= \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}}P_{G_{H1}}^{-\frac{\lambda\epsilon}{1-\lambda}}P_{G_{H2}}^{\frac{-\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon}L_F \\
&= \lambda^{\frac{\lambda}{1-\lambda}}(1-\epsilon)^{\frac{\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\frac{\lambda\epsilon}{1-\lambda}}(\frac{A_{H1}}{\lambda})^{-\frac{\lambda\epsilon}{1-\lambda}}(\frac{A_{H2}}{\lambda})^{\frac{-\lambda(1-\epsilon)}{1-\lambda}}\epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon}L_F
\end{aligned}$$

Then the growth rates of Y_H^{Trade} and Y_F^{Trade} are

$$\frac{\dot{Y_H^{Trade}}}{Y_H^{Trade}} = \Gamma\frac{\dot{Z_{H1}}}{Z_{H1}} + \{\eta[1-(\delta+\gamma)]\epsilon + (\delta+\gamma)(1-\epsilon)\}\frac{\dot{Z_{H2}}}{Z_{H2}} + \{(1-\eta)[1-(\delta+\gamma)]\epsilon\}\frac{\dot{Z_{F2}}}{Z_{F2}} \tag{139}$$

and

$$\frac{\dot{Y_F^{Trade}}}{Y_F^{Trade}} = \Gamma\frac{\dot{Z_{H1}}}{Z_{H1}} - (\delta+\gamma)\epsilon\frac{\dot{Z_{H2}}}{Z_{H2}} + \{[1-(\delta+\gamma)]\epsilon + (\delta+\gamma)\}\frac{\dot{Z_{F2}}}{Z_{F2}} \tag{140}$$

where $\Gamma = 2(\delta+\gamma)\epsilon + 1 - \epsilon - (\delta+\gamma)$, and $\Gamma + [-(\delta+\gamma)\epsilon] + [(1-(\delta+\gamma))\epsilon + (\delta+\gamma)] = 1$. On the BGP, recall from section (3) that c_i/Y_i is constant, so

$$\frac{\dot{Y_H^{Trade}}}{Y_H^{Trade}} = \frac{\dot{Y_F^{Trade}}}{Y_F^{Trade}} = \frac{\dot{Z_{H1}}}{Z_{H1}} = \frac{\dot{Z_{H2}}}{Z_{H2}} = \frac{\dot{Z_{F2}}}{Z_{F2}} = \frac{\dot{C_H}}{C_H} = \frac{\dot{C_F}}{C_F} = r - \rho$$

Combining 128, 129, 130, 131, 133, 134 and 54, we have

$$\begin{aligned} g_{H1} &= \frac{\delta}{1-\delta}\alpha_{H1}\theta_{H1}\frac{Z_{H1}}{\overbrace{Z_{H2}}} - \frac{\rho}{1-\delta} \\ &= \frac{\delta}{1-\delta}\alpha_{H1}\theta_{H1}(\frac{Z_{H1}}{Z_{H2}})^{\eta}(\frac{Z_{H1}}{Z_{F2}})^{1-\eta} - \frac{\rho}{1-\delta} \end{aligned} \tag{141}$$

$$g_{H2} = \frac{\delta}{1-\delta}\alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} - \frac{\rho}{1-\delta} \tag{142}$$

$$g_{F2} = \frac{\delta}{1-\delta}\alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} - \frac{\rho}{1-\delta} \tag{143}$$

By the same reasoning as before, balanced growth implies

$$g_{H1} = g_{H2} = g_{F2} \tag{144}$$

Equating the right sides of 142 and 143, we get

$$\frac{Z_{H2}}{Z_{H1}} = \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}}\frac{Z_{F2}}{Z_{H1}} \tag{145}$$

Substitute that into 141 and equate the right sides of 141 and 143 to get

$$\begin{aligned} \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} &= \alpha_{H1}\theta_{H1}(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}}\frac{Z_{F2}}{Z_{H1}})^{-\eta}(\frac{Z_{H1}}{Z_{F2}})^{1-\eta} \\ &= \alpha_{H1}\theta_{H1}(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{-\eta}(\frac{Z_{H1}}{Z_{F2}}) \\ \Rightarrow (\frac{Z_{H1}}{Z_{F2}})^2 &= \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}}(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{\eta} \\ \Rightarrow (\frac{Z_{H1}}{Z_{F2}})^{\star} &= \sqrt{\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}}(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{\eta}} \end{aligned} \tag{146}$$

Substitute this expression into 145 to get

$$\begin{aligned} (\frac{Z_{H2}}{Z_{H1}})^{\star} &= \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}}(\frac{Z_{F2}}{Z_{H1}})^{\star} \\ &= \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}}\sqrt{[\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}}(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{\eta}]^{-1}} \end{aligned} \tag{147}$$

Substitute 146 and 147 into 141 to get the balanced growth rate under incomplete specialization:

$$g^{Trade\star} = \frac{\delta}{1-\delta}\sqrt{\alpha_{H1}\theta_{H1}(\alpha_{H2}\theta_{H2})^{\eta}(\alpha_{F2}\theta_{F2})^{1-\eta}} - \frac{\rho}{1-\delta} \tag{148}$$

We can compare this growth rate to the rates that prevail under autarky:

$$g_i^{Autarky\star} = \frac{\delta}{1-\delta}\sqrt{\alpha_{i1}\theta_{i1}\alpha_{i2}\theta_{i2}} - \frac{\rho}{1-\delta} \quad \text{where} \quad i = H,\, F$$

Home's growth rate increases if $\alpha_{F2}\theta_{F2} > \alpha_{H2}\theta_{H2}$. Foreign's growth rate increase if $\alpha_{H1}\theta_{H1}(\alpha_{H2}\theta_{H2})^{\eta} > \alpha_{F1}\theta_{F1}(\alpha_{F2}\theta_{F2})^{\eta}$.

Finally, we have

$$(\frac{Z_{H2}}{Z_{F2}})^{\star} = (\frac{Z_{H2}}{Z_{H1}})^{\star} \cdot (\frac{Z_{H1}}{Z_{F2}})^{\star}$$

Substitute this value into 128, 130 and 133. Then noting that all rates of return satisfy $r = g^{\star} + \rho$ on the BGP, we can solve for $N_{H1}^{\star}$, $N_{H2}^{\star}$ and $N_{F2}^{\star}$.

6.6 Transition Dynamics

Recall that from 128, 129, 130, 131, 133 and 134, we have the following results:

$$g_{H1} = \frac{r_{H1}}{\delta} - \alpha_{H1}\theta_{H1}\frac{Z_{H1}}{\widehat{Z_{H2}}} \tag{149}$$

where $\widehat{Z_{H2}} = Z_{H2}^{\eta}Z_{F2}^{1-\eta}$,

$$g_{H2} = \frac{r_{H2}}{\delta} - \alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} \tag{150}$$

$$g_{F2} = \frac{r_{F2}}{\delta} - \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} \tag{151}$$

Within Home, the no-arbitrage condition requires that $r_{H1} = r_{H2}$, as in the closed economy. We get the relation between r_{H1} and r_{F2} in a similar way to we did under complete specialization. Using the growth rates of final goods in both countries, 139 and 140, we get the difference between them:

$$\begin{aligned}\frac{\dot{Y_H^{Trade}}}{Y_H^{Trade}} - \frac{\dot{Y_F^{Trade}}}{Y_F^{Trade}} &= \{\eta[1-(\delta+\gamma)]\epsilon + (\delta+\gamma)(1-\epsilon) + (\delta+\gamma)\epsilon\}\frac{\dot{Z_{H2}}}{Z_{H2}} \\ &\quad + \{(1-\eta)[1-(\delta+\gamma)]\epsilon - [1-(\delta+\gamma)]\epsilon - (\delta+\gamma)\}\frac{\dot{Z_{F2}}}{Z_{F2}} \\ &= \{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\}(\frac{\dot{Z_{H2}}}{Z_{H2}} - \frac{\dot{Z_{F2}}}{Z_{F2}})\end{aligned} \tag{152}$$

where $\{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\} = \eta\epsilon[1-(\delta+\gamma)] + (\delta+\gamma) > 0$ given that $0 < (\eta+\gamma) < 1$.

Recall that in equilibrium, by the Euler equation 54,

$$\frac{\dot{Y_H^{Trade}}}{Y_H^{Trade}} = \frac{\dot{C_H}}{C_H} = r_{H1} - \rho = r_{H2} - \rho$$

and

$$\frac{\dot{Y_F^{Trade}}}{Y_F^{Trade}} = \frac{\dot{C_F}}{C_F} = r_{F2} - \rho$$

By 152,

$$r_{H1} - r_{F2} = \{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\}(\frac{\dot{Z_{H2}}}{Z_{H2}} - \frac{\dot{Z_{F2}}}{Z_{F2}}) \tag{153}$$

Combine $r_{H1} = r_{H2}$ with 149 and 150 to get

$$\begin{aligned}\frac{(\dot{Z_{H1}/Z_{H2}})}{Z_{H1}/Z_{H2}} &= g_{H1} - g_{H2} \\ &= -\alpha_{H1}\theta_{H1}(\frac{Z_{H1}}{Z_{H2}})^{\eta}(\frac{Z_{H1}}{Z_{F2}})^{1-\eta} + \alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} \end{aligned} \tag{154}$$

Now combine 153 with 150 and 151 to get

$$\begin{aligned}\frac{(\dot{Z_{H2}/Z_{F2}})}{Z_{H2}/Z_{F2}} &= g_{H2} - g_{F2} \\ &= \frac{r_{H2} - r_{F2}}{\delta} - \alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} \\ &= \frac{\{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\}}{\delta}(\frac{\dot{Z_{H2}}}{Z_{H2}} - \frac{\dot{Z_{F2}}}{Z_{F2}}) \\ &\quad -\alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}\end{aligned}$$

Which can be rearranged as

$$\begin{aligned}-\alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}} &= \{1 - \frac{\{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\}}{\delta}\}\frac{(\dot{Z_{H2}/Z_{F2}})}{Z_{H2}/Z_{F2}} \\ &= \frac{\{-\eta\epsilon + \eta(\delta+\gamma)\epsilon - \gamma\}}{\delta}\frac{(\dot{Z_{H2}/Z_{F2}})}{Z_{H2}/Z_{F2}} \\ \Rightarrow \frac{(\dot{Z_{H2}/Z_{F2}})}{Z_{H2}/Z_{F2}} &= \frac{\delta}{-\eta\epsilon(1-\delta-\gamma) - \gamma}[-\alpha_{H2}\theta_{H2}\frac{Z_{H2}}{Z_{H1}} + \alpha_{F2}\theta_{F2}\frac{Z_{F2}}{Z_{H1}}]\end{aligned} \tag{155}$$

Define $u \equiv Z_{H1}/Z_{H2}$, $v \equiv Z_{H1}/Z_{F2}$, and $w \equiv Z_{H2}/Z_{F2}$. Note that $v = u \cdot w$. Rewrite 154 and 155 as

$$\begin{aligned}\frac{\dot{u}}{u} &= -\alpha_{H1}\theta_{H1}u^{\eta}v^{1-\eta} + \alpha_{H2}\theta_{H2}u^{-1} \\ &= -\alpha_{H1}\theta_{H1}u^{\eta}(uw)^{1-\eta} + \alpha_{H2}\theta_{H2}u^{-1} \\ &= -\alpha_{H1}\theta_{H1}uw^{1-\eta} + \alpha_{H2}\theta_{H2}u^{-1}\end{aligned}$$

and

$$\begin{aligned}\frac{\dot{w}}{w} &= \frac{\delta}{-\eta\epsilon(1-\delta-\gamma) - \gamma}(-\alpha_{H2}\theta_{H2}u^{-1} + \alpha_{F2}\theta_{F2}v^{-1}) \\ &= \frac{\delta}{\eta\epsilon(1-\delta-\gamma) + \gamma}[\alpha_{H2}\theta_{H2}u^{-1} - \alpha_{F2}\theta_{F2}(uw)^{-1}]\end{aligned}$$

These equations yield

$$\dot{u} = -\alpha_{H1}\theta_{H1}u^2w^{1-\eta} + \alpha_{H2}\theta_{H2} \tag{156}$$

and

$$\dot{w} = \frac{\delta}{\eta\epsilon(1-\delta-\gamma) + \gamma}(\alpha_{H2}\theta_{H2}\frac{w}{u} - \alpha_{F2}\theta_{F2}\frac{1}{u}) \tag{157}$$

as the differential equations for u and w that only depend on u and w. They are non-linear, so we linearize them by a Taylor expansion around steady state u^* and w^*. From section (6.5), the values of the steady states are

$$u^{\star} \equiv (\frac{Z_{H1}}{Z_{H2}})^{\star} = (\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{-1}\sqrt{\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}}(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{\eta}} \tag{158}$$

$$w^\star = (\frac{Z_{H2}}{Z_{F2}})^\star = \frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}} \tag{159}$$

The Taylor expansion for 156 is

$$\dot{u} = -2\alpha_{H1}\theta_{H1}u^\star(w^\star)^{1-\eta}(u-u^\star) - \alpha_{H1}\theta_{H1}(1-\eta)(u^\star)^2(w^\star)^{-\eta}(w-w^\star) \tag{160}$$

Thus the locus of $\dot{u}=0$ is

$$u = (-\frac{1-\eta}{2}\frac{u^\star}{w^\star})w + (3-\eta)u^\star \tag{161}$$

The Taylor expansion for 157 is

$$\begin{aligned} \dot{w} &= \frac{\delta}{\eta\epsilon(1-\delta-\gamma)+\gamma}\alpha_{H2}\theta_{H2}\frac{w-w^\star}{u^\star} + \frac{\delta}{\eta\epsilon(1-\delta-\gamma)+\gamma}(\alpha_{H2}\theta_{H2}w^\star & (162)\\ &\quad -\alpha_{F2}\theta_{F2})(-1)(u^\star)^{-2}(u-u^\star) & (163)\\ &= \frac{\delta}{\eta\epsilon(1-\delta-\gamma)+\gamma}\alpha_{H2}\theta_{H2}\frac{w-w^\star}{u^\star} & (164)\end{aligned}$$

where the second line follows from $w^\star = (Z_{H2}/Z_{F2})^\star = \alpha_{F2}\theta_{F2}/\alpha_{H2}\theta_{H2}$, so $\alpha_{H2}\theta_{H2}w^\star - \alpha_{F2}\theta_{F2} = 0$. The equilibrium locus $\dot{w}=0$ is

$$w = w^\star \tag{165}$$

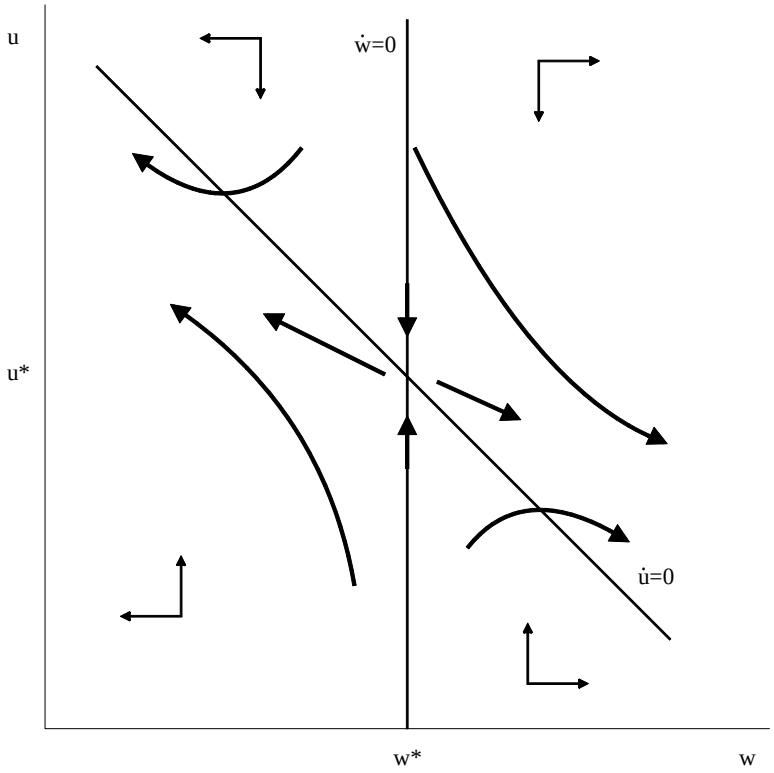

Figure 1: w-u phase diagram

6.7 Stability of Incomplete Specialization

Recall that the trade pattern condition under incomplete specialization is

$$[\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda} \geq \frac{A_{H2}/Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{A_{F2}/Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} = P_{Y_F} > \frac{A_{H1}/Z_{H1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}{A_{F1}/Z_{F1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}$$

and $w \equiv Z_{H2}/Z_{F2}$. Thus under incomplete specialization

$$w \geq \{\frac{[\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda}}{\frac{A_{H2}}{A_{F2}}}\}^{\frac{-\lambda}{(\delta+\gamma)(1-\lambda)}} \tag{166}$$

We have the following results:

1. If $w = w^\star$, then u converges to $u^\star$, and the world economy converges to BGP and stay on incomplete specialization.

2. If $w < w^\star$, then $\dot{w} < 0$. After certain time, the magnitude of w violates 166, and the whole economy evolves into complete specialization. And the complete specialization is stable as we proofed.

3. If $w > w^\star$, then $\dot{w} > 0$. World economy stays on incomplete specialization, but never converges to BGP. In this case, the difference of the growth rates of two countries converges to the constant

$$(\delta + \frac{\delta^2}{\eta\epsilon(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2}\frac{1}{u^\star}$$

where

$$u^\star = (\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{-1}\sqrt{\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H1}\theta_{H1}}(\frac{\alpha_{F2}\theta_{F2}}{\alpha_{H2}\theta_{H2}})^{\eta}}$$

To see this last result, note that from 152 and 157

$$\begin{aligned}
\frac{\dot{Y_H^{Trade}}}{Y_H^{Trade}} - \frac{\dot{Y_F^{Trade}}}{Y_F^{Trade}} &= \{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\}(\frac{\dot{Z_{H2}}}{Z_{H2}} - \frac{\dot{Z_{F2}}}{Z_{F2}}) \\
&= \{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\}\frac{\dot{w}}{w} \\
&= \{\eta\epsilon - \eta(\delta+\gamma)\epsilon + (\delta+\gamma)\}\frac{\delta}{\eta\epsilon(1-\delta-\gamma)+\gamma}\alpha_{H2}\theta_{H2}\frac{w-w^\star}{u^\star}\frac{1}{w} \\
&= (\delta + \frac{\delta^2}{\eta\epsilon(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2}\frac{1}{u^\star} \\
&\quad -(\delta + \frac{\delta^2}{\eta\epsilon(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2}\frac{w^\star}{w} \\
\longrightarrow &\quad (\delta + \frac{\delta^2}{\eta\epsilon(1-\delta-\gamma)+\gamma})\alpha_{H2}\theta_{H2}\frac{1}{u^\star} \quad \text{as} \quad w \to \infty
\end{aligned}$$

6.8 Comparison with the Classical Ricardian Model

We can compare our comparative advantage result with that in the classical Ricardian model, explained by Dornbush, Fisher and Samuelson (1977), denoted hereafter as DFS (1977). We see several differences.

First, the classical Ricardian model does not have the externality discussed in section (5.4), so trade always has a positive effect on output level in the classical Ricardian model. In contrast, in our model the externality makes the level effect uncertain, as explained in section (5.4).

Second, DFS (1977), and most other discussion of Ricardian models, ignore the possibility of incomplete specialization. DFS (1977)'s definition of comparative advantage is similar to ours. DFS (1977) use a multiple good model, so for comparison with our model, we can set the number of goods to 2. Labor is the only factor used to produce trading goods. The wages in Home and Foreign are w_H

and w_F, respectively.[5] The unit labor requirements of good 1 in Home and Foreign areA_{H1} and A_{F1}, respectively. The requirements for good 2 are A_{H2} for Home and A_{F2} for Foreign.[6] Home produces the good with lower domestic labor cost. DFS assume perfect competition, so price equals marginal cost. Home produces good 1 if

$$w_H A_{H1} < w_F A_{F1} \tag{167}$$

which is equivalent to equation (2) in DFS(1977). Trade requires

$$w_H A_{H1} < w_F A_{F1} \quad \text{and} \quad w_H A_{H2} > w_F A_{F2} \tag{168}$$

or, of course, the opposite, but here we focus on the case in 168. That condition can be rewritten as

$$\frac{w_H}{w_F} < \frac{A_{F1}}{A_{H1}} \quad \text{and} \quad \frac{A_{F2}}{A_{H2}} < \frac{w_H}{w_F}$$

The two parts can be combined as

$$\frac{A_{F2}}{A_{H2}} < \frac{w_H}{w_F} < \frac{A_{F1}}{A_{H1}} \tag{169}$$

This last pair of inequalities means that the factor price ratio must be in the interior of the interval of unit-cost ratios. DSF(1977) do not discuss the possibility of corner solutions, where the factor price ratio is on the boundary of the unit-cost ratio interval. Note that 169 implies the comparative advantage expression

$$\frac{A_{H1}}{A_{H2}} < \frac{A_{F1}}{A_F}$$

Trade balance requires

$$\frac{w_H}{w_F} = func(\frac{L_H}{L_F})$$

which is equation (10") and (11) in DFS (1977). Combining this condition with 169, we see that complete specialization requires that the population ratio be inside the interval of the ratios of unit costs:

$$\frac{A_{F2}}{A_{H2}} < func(\frac{L_H}{L_F}) < \frac{A_{F1}}{A_{H1}} \tag{170}$$

If the population ratio is outside the interval specified in 170, one economy will be incompletely specialized.

Third, in the classical Ricardian model, condition 170 is decided by endowment only. If, for example,

$$func(\frac{L_H}{L_F}) < \frac{A_{F2}}{A_{H2}}$$

we have an incompletely specialized world economy, and the world stays that way forever because L_H and L_F are given. The situation is quite different in our model, in which the economy can start in incomplete specialization and evolve into complete specialization.

Our model gives similar results depending on similar conditions, but our trading goods are produced under monopolistic competition rather than perfect competition, our prices are quality-adjusted, our factors of production for the trading goods are endogenous rather than endowed, our model is dynamic, and we also analyze the corner solution of incomplete specialization. In our model, Home chooses the good with lower quality-adjusted price, so the condition for trade is

$$\frac{P_{G_{H1}}}{Z_{H1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \leq \frac{P_{G_{F1}}}{Z_{F1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \quad \text{and} \quad \frac{P_{G_{H2}}}{Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}} \geq \frac{P_{G_{F2}}}{Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}$$

[5] DFS(1977) denote Home and Foreign wages as w and w^*.

[6] We change the notation again to be consistent with our model.

which is similar to the trade condition 168 in the Ricardian model. Complete specialization requires

$$\frac{A_{H2}}{A_{F2}}(\frac{Z_{F2}}{Z_{H2}})^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}} > [\frac{(1-\epsilon)L_H}{\epsilon L_F}]^{1-\lambda} > \frac{A_{H1}}{A_{F1}}(\frac{Z_{F1}}{Z_{H1}})^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}$$

which is similar to 170 in the Ricardian model. Comparative advantage here is given by the ratios of unit costs adjusted by qualities:

$$(\frac{A_{H1}}{Z_{H1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}})/(\frac{A_{H2}}{Z_{H2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}}) \leq (\frac{A_{F1}}{Z_{F1}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}})/(\frac{A_{F2}}{Z_{F2}^{\frac{(\delta+\gamma)(1-\lambda)}{\lambda}}})$$