Learning from Prices: Amplification and Business Fluctuations[*]

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Abstract

We provide a new theory of expectations-driven business cycles in which all shocks are fundamental, prices are fully flexible, and consumers learn from the prices they observe. Learning from prices causes changes in aggregate productivity to shift aggregate beliefs, generating positive price-quantity comovement. The feedback of beliefs into prices can be so strong that arbitrarily small productivity shocks lead to substantial fluctuations. Even with moderate informational feedback, productivity-driven fluctuations take on Keynesian features. Augmented with a public signal about productivity, the model captures several empirical regularities regarding business cycles.

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1 Introduction

We propose a new mechanism—based on learning from prices—that delivers expectations-driven economic fluctuations without relying on any source of extrinsic noise. We show that when consumers learn from the prices of the goods they consume, higher prices can lead consumers to become unduly optimistic about their economic prospects. Initial optimism causes consumers to demand more goods, further increasing prices beyond their full-information level. The self-reinforcing nature of this feedback loop leads to equilibria in which small shocks to supply drive large changes in beliefs, inducing the type of aggregate comovement typically associated with demand shocks.

We demonstrate the potential of this mechanism in a model in which prices are flexible, markets are competitive and the only source of aggregate uncertainty is productivity. Because of learning from prices, our model can replicate several well-known facts about business cycles. These facts are summarized in Table 1: (1) output, inflation, and hours comove; (2) total factor productivity and hours are negatively correlated; (3) inflation is only weakly correlated with output; and (4) aggregate productivity is only weakly correlated with any endogenous aggregate variable. Without learning from prices, productivity shocks would move aggregate prices and quantities in opposite directions, so that matching these facts would require some combination of price-setting frictions, aggregate demand shocks, or exogenous coordination of beliefs on an extrinsic shock.

We study a static microfounded economy inspired by the large family metaphor of Lucas (1980). The economy is inhabited by a representative family whose members are located on islands. On each island live two types of family members who act in the interest of the household. Worker-producers combine local labor with an homogenous factor—capital—to produce a local variety of consumption good. Consumers, in turn, use family resources to buy the local consumption good. The only market connection between islands is the market for capital which is freely traded across islands.

There are two sources of randomness in the economy: one local and one global. First, each island is hit by an idiosyncratic shock that has the effect of a local wealth shock. Second, the production of worker-producers is subject to a common (aggregate) productivity shock.

Table 1: Business Cycle Comovements

	gdp	**hours**	**inflation**	**tfp**
$\rho(\text{gdp}, x)$	1.00	0.86	0.18	-0.06
$\rho(\text{tfp}, x)$	-0.06	-0.36	-0.24	1.00

Note: Data are real per-capita gross domestic product, real per-capita hours in the non-farm business sector, GDP deflator growth, and capacity utilization adjusted TFP described by Basu, Fernald, and Kimball (2006) and maintained by John Fernald at www.frbsf.org. All data are in log-levels, HP-detrended using the longest available sample and smoothing parameter $\lambda = 1600$. Date range: 1960Q1 to 2012Q4.

Worker-producers behave as if they had full information and their choices generate a price for the local good that reflects both local and aggregate exogenous conditions, as well as the aggregate price of capital, which is endogenous to the equilibrium actions of agents in the economy.

Consumers, in turn, do not observe the local shock when they shop for the local consumption good and must infer it from the prices they see on the market. Consumers are uncertain whether a rising local price is a sign of improving local conditions or of falling aggregate productivity. When aggregate productivity shocks have sufficiently small variance, consumers attribute price increases primarily to improving local conditions, leading price increases to drive demand *up*. Since local shocks average to zero across islands, aggregate productivity shocks are the sole driver of average expectations, coordinating optimistic beliefs precisely when productivity is falling. In general equilibrium, the initial increase in demand raises the price of the capital good, which in turn further pushes up the price of local goods, reinforcing consumers' initial mistaken inference. Learning through prices thus leads to productivity-driven shifts in demand in which prices and quantities move together, while the feedback of aggregate conditions into local prices offers the potential for strong amplification.

When the local consumption price depends strongly enough on aggregate conditions, the feedback of actions into beliefs leads some equilibria to exhibit sizable aggregate fluctuations, even in the limit of arbitrarily small aggregate productivity shocks. Fluctuations occur in the limit because, as aggregate shocks decrease in variance, local price signals better reflect local conditions, increasing the weight that consumers place on their price observations. To an econometrician, the fluctuations emerging at the limit of no aggregate shocks would appear to be driven by sentiment, but the origin of sentiment is different from that described by

Angeletos and La'O (2013) or Benhabib et al. (2015).[1] First, sentiments emerge in our model as a case of extreme sensitivity to fundamental shocks, rather than as a response to extrinsic randomness. Second, our model demonstrates how the price system leads markets to endogenously coordinate on this particular shock to drive beliefs, rather than assuming coordination on the shock from the outset.

Although the model economy may exhibit arbitrarily large amplification, as well as equilibrium multiplicity, neither is necessary for it to deliver fluctuations with Keynesian features. In particular, we show that *all* equilibria imply positive price-quantity comovement in response to productivity shocks of sufficiently small variance. Intuitively, the aggregate comovement of prices and quantities arises because aggregate demand curves become upward sloping. When aggregate shocks are relatively small, and price signals strongly reflect local conditions, agents respond to prices more for the information they convey than for the costs that they impose. A higher price thus leads agents to become sufficiently optimistic about local conditions that they increase, rather than decrease, their demand.

To match the qualitative business cycle facts summarized in Table 1, we extend our basic framework by allowing a portion of productivity to be anticipated by agents in the form of public news. More public information might be expected to mitigate the expectational errors of agents, dampening demand-side effects; on the contrary, a smaller contribution of surprise productivity to the price signal leads agents to place more weight on it, exacerbating price-quantity comovement. At the same time, news about productivity leads aggregate expectations to be correlated not with total realized productivity, but with its unanticipated component, while the anticipated component of productivity delivers typical supply-side movements. As a consequence, the economy appears to be driven by two types of shock, one that reflects supply-side conditions, originating in the predicted component of productivity, and one that reflects demand conditions, originating in the surprise component productivity. With distinct transmission mechanisms for the two components of productivity, the stylized model can generate the qualitative pattern of comovements, magnitudes as well signs, listed in Table 1.

[1] Indeed, we show that these limit equilibria have *exactly* the same stochastic properties as the equilibrium documented by Benhabib et al. (2015).

We discuss several extensions that demonstrate the robustness of the basic insight. First, we show that our analysis easily generalizes to the introduction of noisy private signals about local conditions. We then present a monetary version of our model, microfounding local shocks as shocks to the endowment of nominal wealth held on each island. Next, we show how preferences—specifically, concavity in the disutility of labor—can strengthen the informational amplification that arises in equilibrium. We then prove that the equilibria we emphasize in this paper could be learned by agents inhabiting our economy for sufficiently many periods. Finally, we show that while high prices do indeed spur total demand, the model need not imply the existence of a positive price-quantity relationship at the good level.

Our analysis unifies two competing approaches to rationalize large fluctuations in economic outcomes with the small measured volatility of total factor productivity and other aggregate fundamentals that may drive these outcomes. First, it shares the insight of the recent noise-shock and sentiment literature, which shows that fluctuations may be driven by expectational errors that are correlated across agents (Lorenzoni, 2009; Angeletos and La'O, 2013; Benhabib et al., 2015). Second, it shares the focus on amplification with studies of aggregate transmission mechanisms that lead otherwise modest economic shocks to have large aggregate consequences (Kiyotaki and Moore, 1997; Bernanke et al., 1999; Brunnermeier and Sannikov, 2014). In our environment, expectational errors originate with fundamental shocks and are amplified by agents' inferences using endogenous price signals. This paper is also related to an earlier strand of work seeking amplification mechanisms, often through production externalities, that are strong enough to support sunspot fluctuations (see Azariadis, 1981; Cass and Shell, 1983; Cooper and John, 1988; Manuelli and Peck, 1992; and Benhabib and Farmer, 1994, among others.)

This paper belongs to a long literature that studies coordination games with incomplete information. Amador and Weill (2010), Manzano and Vives (2011), and Vives (2012) show cases in which endogenous private signals can generate multiple equilibria. Venkateswaran (2013) describes how dispersed information can generate amplification in a labor search model. The mechanisms explored by these authors rely on complementarity or substitutability of actions across agents. In our environment, strong amplification arises not because of strategic interactions in pay-offs but due to the information externality embedded in the price signal. Recently, Gaballo (2016) has shown that information transmitted by prices can generate learnable dispersed-information equilibria in the limit of zero cross-sectional variance of fundamentals, for cases in which a distinct non-learnable perfect-information equilibrium also exists. The literature on price revelation in auction markets following Milgrom (1981) also features a dual informational/allocative role for prices. For recent examples, see Rostek and Weretka (2012); Lauermann et al. (2012); Atakan and Ekmekci (2014).

Recent work by Bergemann and Morris (2013) characterizes the full set of incomplete-information equilibria in similar coordination games. Related work by Bergemann et al. (2015) studies the exogenous information structures that give rise to maximal aggregate volatility, and the extrema they find are typically achieved when the price signal delivers sentiment-like fluctuations in our economy. Recent studies by Hassan and Mertens (2011, 2014) have shown that arbitrarily small deviations from rational expectations can generate nontrivial aggregate consequences, in a manner that resembles the multiplier effect that we find. Here, we restrict ourselves to endogenously-generated signals and to rational expectations.

2 A microfounded model

In this section, we present a model with the aim of providing a simple and transparent intuition of our main mechanism. Although stylized, our economy gives full microfoundations to the information structure that generates imperfect learning. In particular, all shocks are fundamental in nature and all signals are derived as endogenous outcomes of competitive markets.

2.1 Preferences and technology

The economy is inhabited by a representative price-taking household composed of a continuum of members inhabiting different islands. Each member can be either a consumer or a worker-producer.[2] Members are evenly distributed across islands indexed by $i \in [0,1]$. On

[2] A similar family structure has been used previously by Lucas (1980) and more recently by Amador and Weill (2010) and Angeletos and La'O (2010). The distinct roles of family members allows us to focus on

each island i, a representative worker-producer produces a local consumption good of variety i employing labor of type i and an homogeneous productive input, capital, while a representative consumer buys consumption goods of variety i.[3] Each individual agent operates to maximize the household utility but is informationally isolated; agents cannot pool their information across islands or across agent types.

The utility function of the family is:

$$\int e^{\mu_i} \left(\log C_i - \phi N_i\right) di, \tag{1}$$

where C_i and N_i denote, respectively, consumption and labor of variety i, ϕ is a positive constant, and e^{μ_i} is an local shock with $\mu_i \sim N(0, \sigma_\mu)$ independently distributed across islands. The local shock captures changes in the relative value of each island's contribution to family utility. The effect of the local shock is isomorphic to a local wealth shock, as it simultaneously increases the appetite for consumption and decreases the desire to work on a specific island relative to other islands. In a monetary extension of this model, presented in section 5.2, we show that local shocks can be modeled as local variations in the stock of money held in each island.

The household pools together resources from all the islands and provides funds to consumers to buy the local variety. The household budget constraint is

$$\int P_i C_i di = QZ + \int W_i N_i di + \int \Pi_i di, \tag{2}$$

where P_i is the price of the good i, W_i is the nominal wage of labor type i, Π_i is the profit in island i, and Q is the price of the capital good, which is available in a fixed supply Z and trades freely across islands. Finally, as in Angeletos and La'O (2013), we normalize the value of the Lagrange multiplier associated with the budget constraint of the household to serve as a numeraire.[4]

the signaling role of consumption prices and to avoid confounding effects that might arise, for example, if households also learn form their experience in labor markets.

[3]While we call the input good capital, our mechanism only requires the existence of some input whose aggregate supply is predetermined within the period.

[4]We could have equivalently fixed the average wage to one, as do Benhabib et al. (2015). In the appendix, we show that our normalization is equivalent to fixing a monetary numeraire, which is the typical approach in the DSGE literature; our economy can be seen as the "cashless" limit of a monetary economy. We could also have obtained the same result in an i.i.d. dynamic economy by allowing the household to trade a nominal

The tradable capital good is combined with island-specific labor to produce the final good, C_i, according to the technology,

$$C_i = N_i^{\gamma} \left(e^{\zeta} Z_{(i)}\right)^{1-\gamma}, \tag{3}$$

where $\gamma \in (0,1)$ is the labor share in the economy, $Z_{(i)}$ denotes the quantity of the capital good used in the production of consumption good i, and e^{ζ} is an aggregate productivity shock distributed as $\zeta \sim N(0, \sigma_{\zeta})$.

In our baseline information structure, we assume that worker-producers have full information while consumers do not. Instead, consumers must infer local conditions from their observation of the local price, P_i. The motivation for this modeling choice is the fact that, in actual economies, consumers generally devote fewer resources to information acquisition than do producers, who must gather information professionally to plan production and trade on the input market. Nevertheless, this asymmetry in information need not be considered a primitive of the economy. In particular, we can imagine that both worker-producers and consumers enter their markets with dispersed priors of equal precision about the state of the economy. Since worker-producers trade on both a local market and a global market, they will perfectly discern local and global conditions. Since consumers interact only in their local market, however, their market experiences will not allow them to do the same.

In later sections, we generalize our baseline structure. In Section 4, we study the case where consumers have public information about productivity. This extension proves important in reproducing the salient facts of the business cycle. In section 5.1, we show that our baseline analysis case carries over to the case in which the consumer also observes a private signal about the local shock.

We can write the maximization problems of the worker-producer in island i as

$$\text{worker-producer} \quad : \quad \max_{N_i} \; W_i N_i - e^{\mu_i} \phi N_i, \tag{4}$$

$$: \quad \max_{N_i, Z_{(i)}} \; P_i N_i^{\gamma} \left(e^{\zeta} Z_{(i)}\right)^{1-\gamma} - W_i N_i - Q Z_{(i)}, \tag{5}$$

bond in zero net supply and ruling out bubbles.

and the problem of the consumer on the same island as

$$\text{consumer} \quad : \quad \max_{C_i} \; E[e^{\mu_i}|P_i]\log C_i - P_i C_i. \tag{6}$$

The only role for incomplete information arises from consumers' ignorance regarding the exogenous island-specific disturbance, μ_i. She updates her beliefs about this shock from her observation of the price of her good, P_i, which depends on both local conditions and the price of capital, Q, which, in turn, depends on aggregate demand for capital.

The formal definition of equilibrium is given by the following.

Definition 1. *For a given realization of* $\{\mu_i\}_0^1$ *and* ζ*, a rational expectations equilibrium is a collection of prices* $\{\{P_i, W_i\}_0^1, Q\}$ *and quantities* $\{N_i, C_i, Z_{(i)}\}_0^1$ *such that family members' choices are optimal given the prices they observe, and markets clear.*

The important feature of the equilibrium that we are going to study is that family members seek to maximize the same utility function, but are informationally separated and only learn through their market experience. In particular, given the competitive nature of the markets, family members are able to achieve the social optimum (welfare is trivially the family utility) under perfect information. This setting ensures that our results do not hinge on market failures and, instead, arise exclusively from the lack of perfect information.

2.2 Equilibrium with learning from prices

The first-order conditions of the family members' problems are:

$$E[e^{\mu_i}|P_i] = C_i P_i, \tag{7}$$

$$W_i = e^{\mu_i}\phi, \tag{8}$$

$$Q = (1-\gamma)P_i N_i^{\gamma} Z_{(i)}^{-\gamma} e^{(1-\gamma)\zeta}, \tag{9}$$

$$W_i = \gamma P_i N_i^{\gamma-1} Z_{(i)}^{1-\gamma} e^{(1-\gamma)\zeta}. \tag{10}$$

Given our functional form assumptions, the economy admits an exact log-linear solution. Letting $x \equiv \log(X/\bar{X})$ for any level variable X, the full set of equilibrium conditions of the economy can be written using the log-deviations of each variable from its steady-state value

$\bar{X}$. Combining the log-linear version of (9) and (10), we obtain the standard result,

$$p_i = \gamma w_i + (1-\gamma)(q-\zeta), \tag{11}$$

which states that the equilibrium price of the local good is a linear combination of the costs of factor inputs, corrected for productivity, with weights according to the share of that input in production.

From equation (8), it follows that $w_i = \mu_i$, i.e., the wage is a direct measure of the local shock. Combining the optimality condition for z in (9) with the production function in (3), it is possible to show that $q = p_i + c_i - z_{(i)}$. Then, using the log-linear version of consumer optimality in (7) and exploiting the market-clearing condition, $\int z_{(i)} di = 0$, we have

$$q = \int E[\mu_i | p_i] di. \tag{12}$$

Equation (12) states that fluctuations in the price of capital are driven only by the correlated component of consumers' expectations about their own local conditions. We can therefore rewrite the marginal cost expression in (11) as

$$p_i = \gamma \mu_i + (1-\gamma)\left(\int E[\mu_i | p_i] di - \zeta\right). \tag{13}$$

The signal structure implied by this final equation captures the endogenous feedback effect of inference *from* prices back *into* prices, and it is on this structure that we focus our subsequent analysis.

Before proceeding to an analytical characterization, it is helpful to spell out the economic intuition behind the inference problem being solved by consumers. When consumers see the equilibrium price of their good fluctuating, they cannot determine the extent to which the change is due to island-specific rather than economy-wide factors. From equation (13), it is clear that an increase in price can be triggered by local factors—that is, by an increase in the local wage—in which case the higher price indicates an increase in the marginal value of the local variety. Nevertheless, the same increase in price could also be driven by aggregate factors, either an increase in the price of capital or a decrease in aggregate productivity, that are not related to local conditions. Consumers' confusion about these sources of price fluctuations means that a price increase driven by a small negative productivity shock is at least partially interpreted by consumers on each island as a positive local shock, thereby potentially triggering an increase in demand for all local final goods. Higher demand for final goods leads to higher demand for the inelastically supplied capital good, raising its price, which then feeds back and is reflected again in final good prices. The fact that consumers extract information from local prices thus amplifies the volatility of the capital good's price, making consumers' *equilibrium* inference worse.

The following proposition provides a characterization of equilibrium in terms of the profile of expectations, so that it will be easy to map the outcomes of the inference problem to the equilibria of the economy.

Characterization of the equilibrium. *An equilibrium is characterized by a profile of consumers' expectation* $\{E[\mu_i|p_i]\}_{i=0}^{1}$ *so that, given (12), in each island* $i \in (0,1)$ *we have*

$$
\begin{aligned}
p_i &= \gamma\mu_i + (1-\gamma)(q-\zeta), && (14)\\
c_i &= E[\mu_i|p_i] - \gamma\mu_i - (1-\gamma)(q-\zeta), && (15)\\
w_i &= \mu_i && (16)\\
n_i &= E[\mu_i|p_i] - \mu_i, && (17)\\
z_{(i)} &= E[\mu_i|p_i] - q. && (18)
\end{aligned}
$$

A rational expectations equilibrium is one for which consumers' expectations, $E[\mu_i|p_i]$*, are rational.*

Proof. Derivations are provided in Appendix A.1. ∎

It is easy to check that, when consumers have perfect information, price and quantity move in opposite directions.[5] In particular, a positive productivity shock produces a typical-looking supply-driven fluctuation: Total production goes up and the average price level falls.

3 Amplification through learning

In this section, we analyze the signal extraction problem created by the information structure microfounded above. We show how to solve the consumers' inference problem, highlighting

[5]Substitute μ_i for $E[\mu_i|p_i]$, then substitute (14) into (15), and take the integral on both sides to get $c = -p$.

the strategic interaction engendered by the endogeneity of the price signal. In particular, we demonstrate that informational feedback can generate amplification of fundamental shocks, which in some cases is strong enough to deliver nontrivial responses to vanishingly small shocks. All the analysis in this section easily generalizes to the case where consumers also possess private signals about local conditions, as we show in section 5.1.

Best individual weight function

Given her price signal, p_i, consumer i must infer μ_i, the marginal utility of her consumption type. The key feature of the signal extraction problem is that the precision of the signal depends on the nature of average actions across the population and, therefore, on the average reaction of other consumers to their own price signals. A rational expectations equilibrium is a situation in which the individual reaction to the signal is consistent with its actual precision, i.e., is an optimal response to the average reaction of others.

We now characterize the equilibria of the economy. Since we assume that all stochastic elements are normal, the optimal forecasting strategy is linear. As a consequence, the individual expectation is linear in p_i and can be written as

$$E[\mu_i|p_i] = a_i \left(\gamma\mu_i + (1-\gamma) \left(\int E[\mu_i|p_i]di - \zeta \right) \right), \tag{19}$$

where a_i is the coefficient, determined prior to the realization of shocks, that measures the strength of the reaction of consumer i's beliefs to the signal she will receive. Since the signal is *ex ante* identical for all consumers, each uses a similar strategy, and we can recover the average expectation by integrating across the population:

$$\int E[\mu_i|p_i]di = a\,(1-\gamma) \left(\int E[\mu_i|p_i]di - \zeta \right), \tag{20}$$

with $a \equiv \int a_i di$ denoting the average weight applied to the signal. Solving the expression above for the average expectation yields

$$\int E[\mu_i|p_i]di = -\frac{a\,(1-\gamma)}{1-a\,(1-\gamma)}\zeta, \tag{21}$$

which is a nonlinear function of the average weight, a. Importantly, this function features a singularity at the point $1/(1-\gamma)$. When $a < 1/(1-\gamma)$, the average expectation comoves with the productivity shock and the opposite holds when $a > 1/(1-\gamma)$.

The variance of the aggregate expectation—equivalently, of the capital price—is given by

$$\sigma_q^2(a) = \left(\frac{a\,(1-\gamma)}{1-a\,(1-\gamma)}\right)^2 \sigma^2, \tag{22}$$

where $\sigma_q^2 \equiv var(q)/\sigma_\mu^2$ and $\sigma^2 \equiv \sigma_\zeta^2/\sigma_\mu^2$ are the variances of the aggregate expectation and the aggregate shock, respectively, once each is normalized by the variance of the idiosyncratic fundamental.

Substituting the average expectation in (21) into the price signal described in equation (13), we get an expression for the local price exclusively in terms of the idiosyncratic and aggregate shocks:

$$p_i = \gamma\mu_i + \frac{\gamma-1}{1-a\,(1-\gamma)}\zeta, \tag{23}$$

whose precision with regard to μ_i is given by

$$\tau(a) = \left(\frac{\gamma\,(1-a\,(1-\gamma))}{(1-\gamma)\sigma}\right)^2. \tag{24}$$

We are now ready to compute the consumer's optimal inference, taking the average weight of other consumers as given. We seek an a_i such that $\mathrm{E}[p_i(\mu_i - a_i p_i)] = 0$, i.e., the covariance between the signal and forecast error is zero in expectation. This condition implies that information is used optimally. The best individual weight is given by

$$a_i(a) = \frac{1}{\gamma}\left(\frac{\tau(a)}{1+\tau(a)}\right). \tag{25}$$

Given the linear-quadratic environment, we can interpret $a_i(a)$ in a game-theoretic fashion as an individual's best reply to the profile of others' actions summarized by the sufficient statistic a. To be precise, every a_i is associated with one and only one contingent strategy that describes the conditional expectation $E[\mu_i|p_i] = a_i p_i$ of consumer i, where p_i identifies a set of states of the world indistinguishable to consumer i.

Equilibria

Given that agents face an information structure with the same stochastic properties, a rational expectations equilibrium must be symmetric. This last requirement completes our notion of equilibrium, which is formally stated below.

Definition 2. *A rational expectations equilibrium is characterized by a profile of shoppers' expectations* $\{E[\mu_i|p_i]\}_{i=0}^{1}$ *such that* $E[\mu_i|p_i] = \hat{a}p_i$ *with* $a_i(\hat{a}) = \hat{a}$*, for each* $i \in (0,1)$*.*

Our game-theoretic interpretation of the optimal coefficient makes clear the equivalence between a rational expectations equilibrium and a Nash equilibrium: No one has any individual incentive to deviate when everybody else conforms to the equilibrium prescriptions.

An equilibrium of the model is a fixed point of the individual best-weight mapping given by equation (25). In practice, there are as many equilibria as intersections between $a_i(a)$ and the bisector. The fixed-point relation delivers a cubic equation, which may have one or three real roots. The following proposition characterizes these equilibrium points.

Proposition 1. *For* $\gamma \geq 1/2$*, there always exists a unique REE equilibrium for* $\hat{a} = a_u \in (0, \gamma^{-1})$*.*

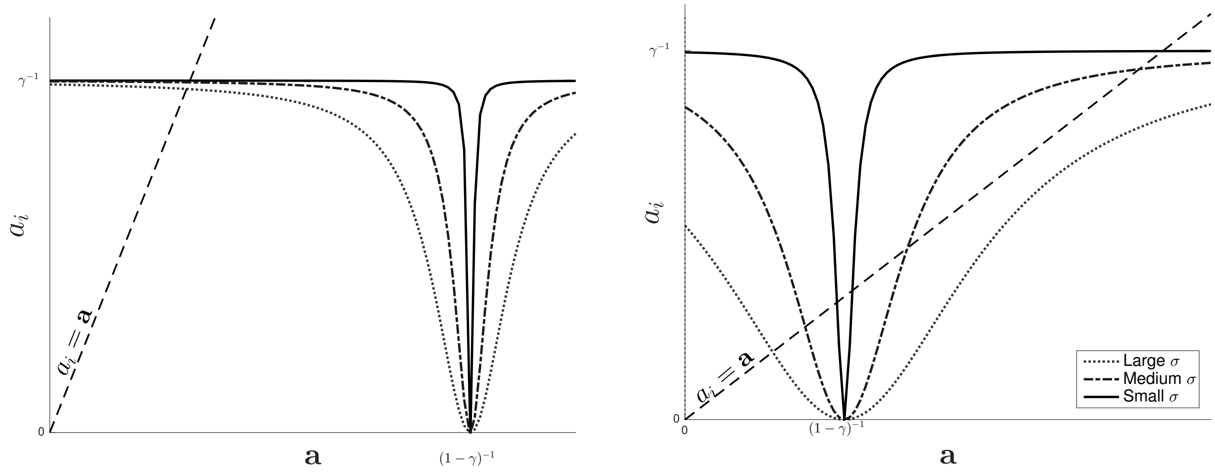
For $\gamma < 1/2$*, there always exists a low REE equilibrium for* $\hat{a} = a_- \in (0, (1-\gamma)^{-1})$*. In addition, there exists a threshold* $\bar{\sigma}^2$ *such that, for any* $\sigma^2 \in (0, \bar{\sigma}^2)$*, a middle and a high REE equilibrium also exist for* $\hat{a} = a_\circ$ *and* $\hat{a} = a_+$*, respectively, both lying in the range* $((1-\gamma)^{-1}, \gamma^{-1})$*.*

Proof. Given in Appendix A.2. ∎

Proposition 1 states that when the aggregate component receives relatively high weight in the signal, the model may exhibit multiplicity. In particular, there are three equilibria whenever $\gamma < 1/2$ and the variance of the productivity shock is small enough; otherwise, a unique equilibrium exists. While an analytical characterization of these equilibria is possible, the expressions are rather complicated. Nevertheless, the relevant properties can be grasped from the reaction functions plotted in Figure 1 (see figure caption).

The slope of the $a_i(a)$ curve at the intersection with the bisector determines the nature of the strategic incentives underlying each equilibrium. Equilibria a_u and a_- are characterized by *substitutability in information*, as the optimal individual weight is decreasing in the average weight, i.e., $a_i'(\hat{a}) < 0$.[6] In contrast, the equilibria $a_\circ$ and a_+ are characterized

[6]See equation (63) in appendix A.2.



(a) $\gamma = 0.75$ (b) $\gamma = 0.25$

Figure 1: The figure illustrates four properties of $a_i(a)$ for given γ and σ: (i) $a_i(0) > 0$; (ii) $a_i'(a) < 0$ for $a \in (0, (1-\gamma)^{-1})$, and $a_i((1-\gamma)^{-1}) = 0$; (iii) $a_i'(a) > 0$ for for $a \in ((1-\gamma)^{-1}, \gamma^{-1})$ and $\lim_{a\to\infty} = \gamma^{-1}$; (iv) $\partial a_i(a)/\partial\sigma \geq 0$.

by *complementarity in information* since $a_i'(\hat{a}) > 0$. In fact, as soon as $a > (1-\gamma)^{-1}$, the higher the a the higher the precision of the signal regarding μ_i, which further pushes up the optimal weight. The emergence of complementarity explains the upward-sloping part of the best-weight function and is key for the existence of multiple equilibria.

While complementarity is essential for generating multiple equilibria, it is neither necessary nor sufficient to imply a strong informational multiplier. To see this, define the multiplier, $\Gamma(\hat{a}) \equiv \sigma_q^2(\hat{a})/\sigma^2$, as the volatility of beliefs relative to the volatility of the shock ζ for some equilibrium point $\hat{a}$. We will say that the economy exhibits *amplifying* informational feedback whenever a fall in the volatility of the exogenous shock leads to an increase in $\Gamma(\hat{a})$, i.e., $\partial\Gamma(\hat{a})/\partial\sigma < 0$, and *dampening* feedback otherwise. The following proposition classifies the equilibria in Proposition 1 according to the type of feedback they generate.

Proposition 2. *The equilibria a_u, a_-, and $a_\circ$ all exhibit amplifying feedback, while the equilibrium a_+ exhibits dampening feedback.*

Proof. Given in Appendix A.2. ∎

The characterization of informational feedbacks as either amplifying or dampening depends on whether the equilibrium value of a gets closer to $(1-\gamma)^{-1}$ as σ shrinks. From Figure 1, it is clear that $a_u, a_\circ$, and a_- feature amplifying feedback, whereas a_+ features dampening feedback. Nevertheless, the feedback effects in $a_\circ$ and a_- are distinct from that in a_u for reasons we discuss in the following section.

3.1 Sentiment equilibria as limit case of strong amplification

Here we show that learning from prices can generate such strong amplification of fundamental shocks that the economy can sustain sizable aggregate fluctuations, even in the limit $\sigma^2 \to 0$. We see this as a new characterization of sentiments-driven fluctuations, which have recently received growing attention in the literature. The intuition for this result is captured by Figure 2, which plots, for each equilibrium, the variance of the average expectation as a function of the volatility of productivity shocks. As σ shrinks, the unique and the high equilibria, namely a_u and a_+, approach infinite precision and no aggregate volatility. In contrast, the middle and the low equilibria $a_\circ$ and a_- converge to finite precision and sizable aggregate volatility.

The plots numerically demonstrate that, as σ goes to zero, the informational feedbacks in the middle and low equilibria grow at a speed that makes the product of the two achieve a finite limit. The following proposition establishes the result formally.

Proposition 3. *In the limit $\sigma^2 \to 0$,*

i. the unique equilibrium (for $\gamma \geq 1/2$) and the high equilibrium (for $\gamma < 1/2$) converge to a point with no aggregate volatility:

$$\lim_{\sigma^2 \to 0} a_{u,+} = \max\left(\frac{1}{\gamma}, \frac{1}{1-\gamma}\right) \qquad \lim_{\sigma^2 \to 0} \sigma_q^2(a_{u,+}) = 0. \tag{26}$$

ii. the low and middle equilibria (for $\gamma < 1/2$) converge to the same point and exhibit non-trivial aggregate volatility:

$$\lim_{\sigma^2 \to 0} a_{\circ,-} = (1-\gamma)^{-1} \qquad \lim_{\sigma^2 \to 0} \sigma_q^2(a_{\circ,-}) = \frac{\gamma(1-2\gamma)}{(1-\gamma)^2}. \tag{27}$$

Proof. Given in Appendix A.2. ∎

In the limit of $\sigma \to 0$, the middle and low equilibria have the same stochastic properties as the sentiment equilibria described by Benhabib et al. (2015), although sentiments in that model are driven by extrinsic shocks. In our economy, sentiment fluctuations are driven

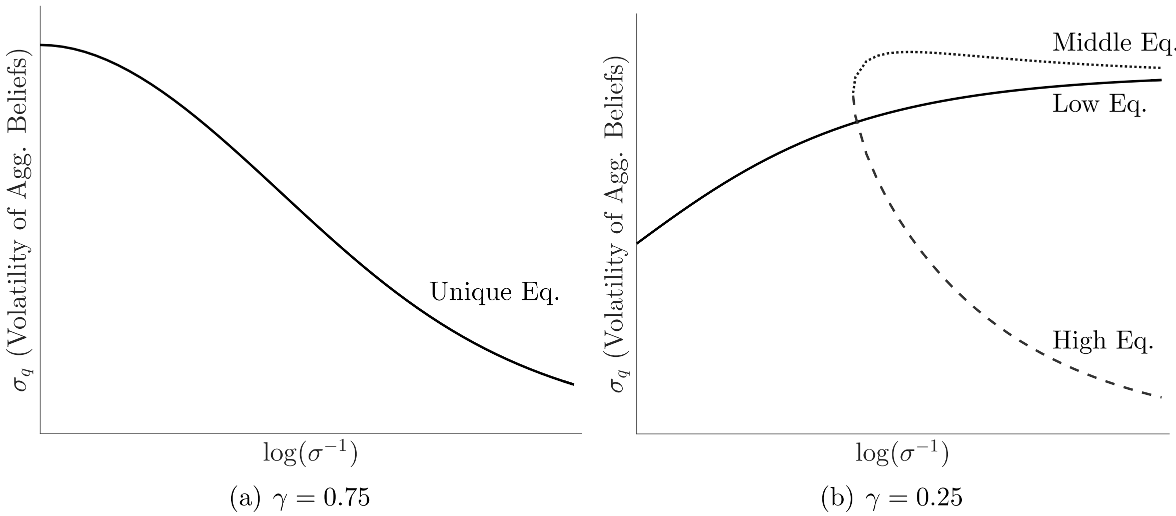


Figure 2: Belief volatility approaching the limit.

by infinitesimally-small fundamental shocks, whose realization is able to coordinate sizable fluctuations in agents' expectations via their effects on the endogenous price signal.

The limiting result suggests that a strict dichotomy between fundamental and non-fundamental fluctuations is misleading. Since endogenous signal structures can generate strong multiplier effects on small shocks, they can deliver fluctuations that effectively span a continuum from purely fundamental-driven to purely sentiment-driven. Of course, this possibility does not preclude the existence of fluctuations that originate from truly payoff-irrelevant shocks, but the possibility of fundamental-based sentiments may appeal to those who find fluctuations driven by coordination on truly payoff-irrelevant shocks implausible.

Moreover, in our economy, agents' coordination on small fundamental shocks as the drivers of beliefs arises endogenously through the competitive price system, rather than being assumed from the outset. The analysis of Benhabib et al. (2015) occurs at the limit point rather than approaching it, so it cannot explain the origins of coordination on a particular sentiment shock.

A final implication of our basic analysis here is that the addition of a small amount of *aggregate* noise in the signal—in this case, captured by the effect of productivity on the price signal—can sustain additional equilibria that do not arise under full information. A previous literature has demonstrated cases in which adding *idiosyncratic* noise to signals can either eliminate (Morris and Shin, 1998) or generate (Gaballo, 2016) additional equilibria. But this is the first time it has been observed, to our knowledge, that adding aggregate noise can cause equilibria to proliferate.

4 Business Cycle Fluctuations

In this section, we explore the implications of the learning-from-prices mechanism for the business cycle comovement of our economy. The central theme of this section is that the mechanism that opens the way to extreme amplification and multiple equilibria is also active in the case of equilibrium uniqueness. Even with less extreme amplification, the same qualitative forces emerge as aggregate productivity shocks become less volatile: agents react to prices more for the information they convey than for the costs they impose, leading unexpected technology shocks to have very different implications than they do under full information. We show how our setting can be extended to allow productivity shocks to generate the appearance of both supply and demand driven fluctuations, thereby bringing the model closely in line with our motivating business cycle facts.

Equilibrium with public news

We begin by extending our framework to include public information in the form of news on productivity. We assume that the productivity shock is composed of two independently distributed components

$$\zeta = \hat{\zeta} + \tilde{\zeta};$$

with $\hat{\zeta} \sim (N, \hat{\sigma}_\zeta^2)$, $\tilde{\zeta} \sim (N, \tilde{\sigma}_\zeta^2)$ and $\hat{\sigma}_\zeta^2 + \tilde{\sigma}_\zeta^2 = \sigma_\zeta^2$. The first term, $\hat{\zeta}$, is public information; it corresponds to a "news" shock or the forecastable component of productivity, and is known to all agents before their choices are made. Conversely, $\tilde{\zeta}$ is unknown to consumers and they seek to forecast it using their observation of prices.[7] The decomposition of productivity into a forecastable and surprise component plays two roles in this section. First, it allows us isolate the effects of learning through prices, as the forecasted component of productivity will transmit in the economy as a usual supply-side shock. Second, by combining the responses of the economy to forecasted and surprise productivity shocks, we will be able to generate

[7]Chahrour and Jurado (2016) show that this information structure is equivalent to assuming that agents observe a noisy signal of productivity, $g = \zeta + \vartheta$.

the rich cross-correlation structure seen in the data.[8] For future reference, let $\hat{\sigma}^2 \equiv \hat{\sigma}_\zeta^2/\sigma_\mu^2$, and $\tilde{\sigma}^2 \equiv \tilde{\sigma}_\zeta^2/\sigma_\mu^2$ be the normalized variances of the forecasted and surprise components of productivity respectively.

Only modest modifications are necessary to characterize equilibrium in this general case. Consumers use the forecasted component to refine the information contained in the price signal by "partialing-out" the known portion of productivity. In particular, we can rewrite consumers' expectation as

$$E[\mu_i|p_i] = a_i\tilde{p}_i, \tag{28}$$

where

$$\tilde{p}_i \equiv p_i + (1-\gamma)\hat{\zeta} = \gamma\mu_i + (1-\gamma)\left(\int E[\mu_i|p_i]di - \tilde{\zeta}\right), \tag{29}$$

represents the new signal embodying the information available to the individual consumer, after she has controlled for the effect of $\hat{\zeta}$. It follows that the equilibrium values $\{a_u, a_-, a_\circ, a_+\}$ and the conditions for their existence are isomorphic to the ones in the baseline economy once $\tilde{\sigma}^2$ takes the place of σ^2.

An immediate implication is that increasing the fraction of productivity that is forecastable actually pushes the economy towards a situation of high information multipliers and, when $\gamma < 1/2$, towards the region of equilibrium multiplicity. For the low equilibrium, this implies an *increase* in the variance of the average expectation of consumers. This result demonstrates that the mechanism of Section 3 is robust to increasing the information sets of consumers; so long as any aggregate component remains unknown, agents may endogenously coordinate their errors though the pricing system.

Fact 1: Supply shocks generate demand-driven fluctuations

Our key observation, from the standpoint of generating realistic business cycles, is that all the equilibria of our model can generate business cycle fluctuations with demand-side features; that is, final good prices, total output, the price of capital, and total employment all positively comove in response to the surprise component of productivity. This happens in *all* equilibria because, as aggregate volatility falls, the informational value of the price

[8] Notice that the data presented in Table 1 are about inflation and not price level. In our static setting, the price level and inflation are the same.

signal rises, leading agents' beliefs about their local conditions to respond more strongly to it. Stronger aggregate effects on beliefs eventually lead the informational channel of prices to dominate, so that consumption increases in response to higher prices. In this way, learning from prices provides a new mechanism for generating expectations-driven demand shocks in an economy hit only by fundamental shocks to productivity.

This consequence of endogenous information for business cycle comovements can be seen intuitively by analyzing the aggregate demand and aggregate supply schedules in our economy. Given (3), (7), and (17), we can express aggregate demand and supply as

$$AD \quad : \quad c = q - p, \tag{30}$$

$$AS \quad : \quad c = \gamma q + (1-\gamma)\zeta. \tag{31}$$

When the price of capital q has no effect on consumers' beliefs, this relationship implies a standard downward-sloping aggregate demand relation. However, this changes once we account for the equilibrium feedback of prices into consumers' inference.

To derive equilibrium aggregate demand and supply relations, notice that the average capital price q is a function of both the average price in the economy and the public signal,

$$q = a(p + (1-\gamma)\hat{\zeta}).$$

Substituting this expression into the aggregate demand and aggregate supply expressions in equations (30) and (31) yields

$$AD \quad : \quad c = (a-1)p + a(1-\gamma)\hat{\zeta} \tag{32}$$

$$AS \quad : \quad c = \gamma a p + (1 + a\gamma)(1-\gamma)\hat{\zeta} + (1-\gamma)\tilde{\zeta}. \tag{33}$$

Notice that *both* aggregate demand and aggregate supply are shifted by the forecasted productivity shock, $\hat{\zeta}$, while the surprise component, $\tilde{\zeta}$, shifts only aggregate supply. This is natural since, in our environment, the surprise productivity shock affects consumers' actions only though its effect on prices. It is easy now to see the following.

Proposition 4. *For $\tilde{\sigma}^2$ sufficiently small, all equilibria exhibit comovement of aggregate output, employment, the price level, and the price of capital in response to surprise productivity shocks.*

Proof. The results follows from continuity of the best-response function, and the observation that all limit equilibria entail $\hat{a} > 1$. ∎

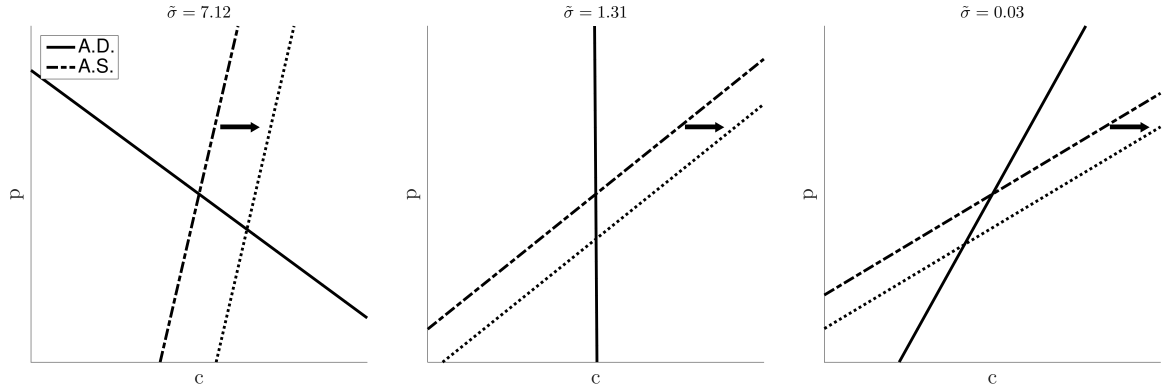
Crucially, the relation in (32) implies that aggregate demand is upward sloping for any a larger than unity. In this case, price and quantity will move together in response to shifts of either aggregate demand or aggregate supply! Moreover, as the relative variance $\tilde{\sigma}$ decreases, this will be true *for all equilibria* in the economy. Even the unique and high equilibria, which display no fluctuations in the limit $\sigma \to 0$, exhibit comovements in prices and quantities away from that limit, *as if* the economy is hit by a common demand shock. In fact, the equilibrium condition $a > 1$ always entails a situation in which the informational content of prices is more important than their allocative effect, that is, movements in expected marginal utility of a good more than compensate for a change in its price. In the model driven by aggregate productivity shocks, the consequences for aggregate demand have immediate implications for the comovement of prices and quantities in the economy.

Figure 3 plots aggregate supply and demand relations for different values of the relative volatility σ, in two cases: one where the economy always has a unique equilibrium ($\gamma = 0.75$) and one where multiplicity is possible ($\gamma = 0.25$), in which case we consider the low equilibrium. As $\tilde{\sigma}$ shrinks, in both cases, the slope of aggregate demand turns clockwise until it becomes upward sloping. In particular, the upward slope in aggregate demand exceeds the slope of aggregate supply (which also turns but much less), as shown by the two panels in the last column. In both equilibria, when the variance of productivity shocks is sufficiently small, outward shifts in supply move prices and quantities in the same direction.

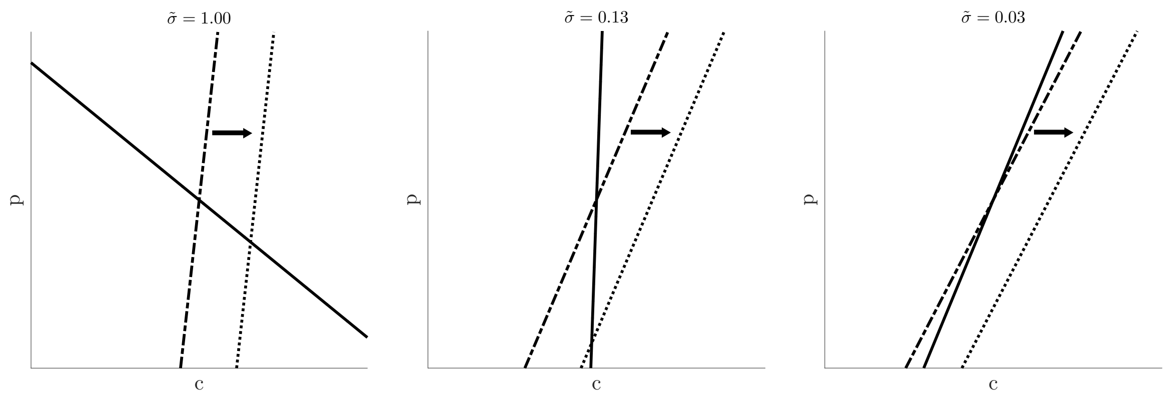
Therefore, aggregate demand in the low equilibrium behaves in a manner that qualitatively resembles its behavior in the unique equilibrium. The peculiarity of the low equilibrium is that, in the limit of $\tilde{\sigma}$ approaching zero, supply and demand overlie each other.[9] The last panel of Figure 3 therefore provides an easy intuition for the extremely large informational multiplier implied by our sentiment-like equilibria, as even small shifts in aggregate supply imply large changes in the equilibrium quantity of consumption.

Although similar in many respects, the curves in Figure 3 also suggest one reason why the

[9] In particular, the limit situation corresponds to $AD : p = (\gamma/(1-\gamma))c$ and $AS : p = c + (1-\gamma)\zeta$ when considering a_u and a_+, for which $\lim_{\sigma\to 0} a_i(a_{u,+}) = 1/\gamma$; $AD : p = ((1-\gamma)/\gamma)c$ and $AS : p = ((1-\gamma)/\gamma)c + ((1-\gamma)^2/\gamma)\zeta$ when considering $a_\circ$ and a_- for which $\lim_{\sigma\to 0} a_i(a_{\circ,-}) = (1-\gamma)^{-1}$.



(a) Unique Equilibrium ($\gamma = 0.75$)



(b) Low Equilibrium ($\gamma = 0.25$)

Figure 3: Aggregate supply and demand in the microfounded model.

unique equilibrium economy looks the most promising for quantitative analysis: the region of $\tilde{\sigma}$ in which upward sloping demand emerges is far larger when $\gamma > 0.5$. Thus, even though fluctuations do disappear as shocks go to zero in the unique equilibrium economy, that version of the model is also able to generate large informational effects of the price signal well away from the limit.

Fact 2: Contractionary Technology

One robust — and from the perspective of an RBC model, surprising — fact about business cycles is that hours typically fall on impact in response to improvements in aggregate technology, while aggregate productivity is only weakly associated with output at any horizon. Basu et al. (2006) document the first fact in detail, and show that it can be rationalized in the context of a sticky price model. Our model offers an alternative account.

To see that hours can fall in response to technology improvements, recall that aggregate labor supply is equal to the average expectation in the economy. By equation (21), we have

$$n = \int E[\mu_i | p_i] di = -\frac{a(1-\gamma)}{1-a(1-\gamma)}\tilde{\zeta}. \tag{34}$$

Thus, unanticipated positive technology shocks lead to a decrease in hours whenever $a < (1-\gamma)^{-1}$, which is always true of the low and unique equilibria.[10] The intuition is straightforward: an increase in the price seen by a consumer could be caused by improving local conditions or by falling aggregate productivity and agents become overly optimistic precisely when (the unobserved part of) productivity is falling. We demonstrate shortly that forecasted productivity shocks have no effect on labor in our economy, implying that the correlation between hours and total productivity always both negative and imperfect.

While the correlation of hours with productivity is unambiguous in the model, the output-productivity relationship is slightly more subtle. One again, it turns out that this relationship hinges on the strength of the learning from prices channel. In particular, surprise productivity shocks cause output contractions in the unique and low equilibria whenever information effects are strong enough, that is whenever $\tilde{\sigma}$ is small enough that aggregate demand slopes upward, an intuition that can again be seen in Figure 3.

Facts 3 and 4: Weakening price-quantity correlation

While Proposition 4 shows that surprise shocks induce positive comovement among business cycle variables, the opposite is true in the case of shocks to productivity that are forecasted. Since forecasted productivity shocks affect both supply and demand, it is helpful to solve for equilibrium consumption and price as a function of shocks and the equilibrium inference coefficient:

$$p = -(1-\gamma)\hat{\zeta} - \frac{(1-\gamma)}{1-a(1-\gamma)}\tilde{\zeta} \tag{35}$$

$$c = (1-\gamma)\hat{\zeta} - \frac{(a-1)(1-\gamma)}{1-a(1-\gamma)}\tilde{\zeta}. \tag{36}$$

From equation (36) it is immediate that forecasted technology shocks always expand

[10] Conversely, it never holds for the high and middle equilibria.

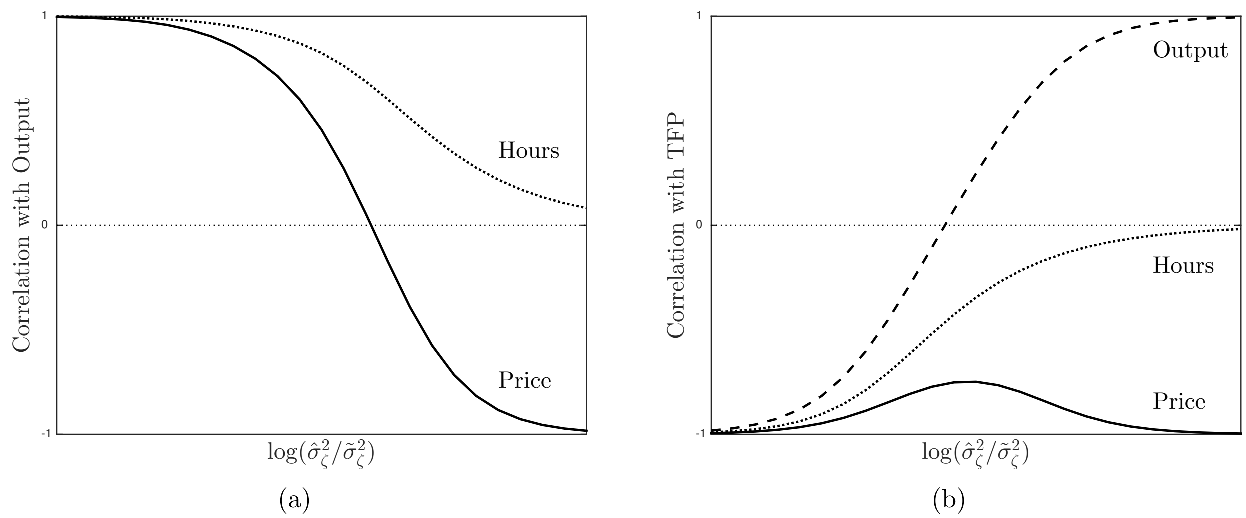


Figure 4: Correlations in the economy with both anticipated and unanticipated technology, with $\gamma = 0.55$ and $\sigma = 0.1$.

output, while equation (34) implies zero impact on labor supply. Moreover, comparing (35) and (36), it is clear that the foreseen component of ζ will move prices and quantities in opposite directions, generating the comovement more typically associated with a supply shock. Thus, overall comovements — the degree to which labor and prices are procyclical, as well as the correlation of total factor productivity with all endogenous variables — will depend on the balance of forecastable and surprise productivity, as well as the overall size of these shocks relative to local conditions.

Business cycle: all facts together in the unique equilibrium

Putting together the observations above, it is plain that our model can qualitatively match our set of business cycle facts one at a time, but can it match them simultaneously? It turns out the answer is yes! The key degree of freedom, and the only one we exploit here, is the decomposition of productivity into its news and surprise components. Because the economy responds differently to these components, we can combine the demand-like effects of surprise productivity shocks with the supply effects of forecasted productivity shocks, delivering comovements between minus one and one.

Figure 4 plots the correlations of output, prices, hours, and productivity as function of the ratio $\hat{\sigma}_\zeta/\tilde{\sigma}_\zeta$ (while fixing their sum) in the unique equilibrium economy with $\gamma = 0.55$. When only a small fraction of productivity is forecastable, comovements are driven by the strong

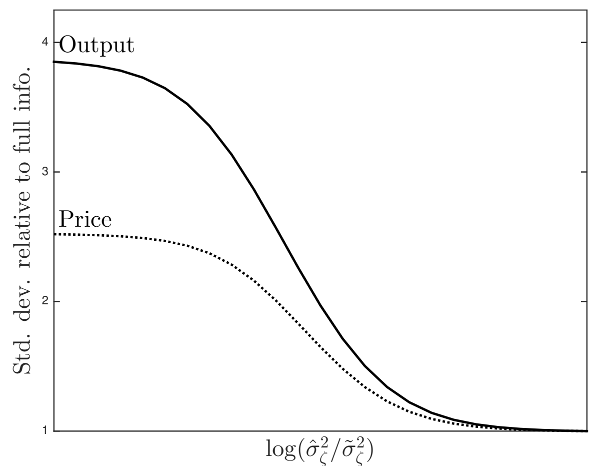


Figure 5: Aggregate volatility with $\gamma = 0.55$ and $\sigma = 0.1$.

information effects inherent in surprise shocks, leading to strongly positive price-quantity comovements, perfectly contractionary productivity shocks, and a price level that is very strongly negatively correlated with TFP. Conversely, in the extreme of perfectly forecasted productivity, the economy appears to be driven by pure supply shocks, with perfect negative correlation of output and prices. In the intermediate range of this ratio, however, these forces offset each other, leading to correlations that qualitatively match all of the implications in Table 1: hours are strongly procyclical, prices are procyclical but less strongly, hours are negatively correlated with productivity while output is only weakly correlated with it, and prices are imperfectly negatively correlated with productivity. While simple, the model does a remarkable job matching the stylized facts with which we began.

Finally, Figure 5 plots the overall degree of amplification in the unique equilibrium economy as a function of the fraction of shocks that are forecasted. As suggested by equation (36), the overall size of the response to surprise shocks is substantially larger than to forecasted productivity shocks, such that in the extreme of perfectly unforecastable technology, output is roughly four times as volatile as it is under full information. Prices, while amplified, are only about 2.5 times as volatile as under full information. In the intermediate range that best matches the various business cycle moments in Table 1, the figure shows overall output volatility that is roughly double that implied by the model under full information. In short, even when the economy has a unique equilibrium, the model delivers substantial amplification of aggregate productivity shocks.

5 Extensions and Discussion

This section presents several extensions to the basic setup, showing that the insights of the main mechanism are robust to various modeling details. First, we provide an alternative microfoundation for the local shocks as shocks to nominal wealth. We then show that the case where consumers have additional private information about local conditions maps into the analysis of Section 3 as a case with a larger variance of productivity shocks. Next, we show that convexity in the disutility of labor (i) expands the range of γ for which strong informational multipliers and equilibrium multiplicity may arise, and (ii) induces wages to comove positively along with prices and quantities for sufficiently small values of σ. We then study the issue of stability under adaptive learning and demonstrate that the low, the high and the unique equilibrium are always learnable, whereas the middle equilibrium is never learnable. Finally, we allow for the disaggregation of goods at the island level to demonstrate that the existence of upward-sloping aggregate demand in our model does not require the existence of Giffen-type goods at the micro level.

5.1 Signal extraction problem with private signals

Here we show how the signal extraction problem extensively studied above is modified by the availability of a private signal about the local shock. The addition of private information maps into our analysis of Section 3 as an increase in the relative variance of aggregate shocks.

Let us assume that a consumer $j \in (0,1)$ in island i has a private signal

$$\omega_{i,j} = \epsilon_i + \eta_{i,j} \tag{37}$$

where $\eta_{i,j} \sim N(0, \sigma_\eta)$ is identically and independently distributed across consumers and islands. In this case, consumers form expectations according to

$$E[\epsilon_i|p_i, \omega_{i,j}] = a\left(\gamma\epsilon_i + (1-\gamma)\left(\int E[\epsilon_i|p_i, \omega_{i,j}]di - \zeta\right)\right) + b\left(\epsilon_i + \eta_{i,j}\right),$$

where b measures the weight given to the additional private signal. Averaging out the relation above and solving for the aggrgeate expectation gives

$$\int E[\epsilon_i|p_i,\omega_{i,j}]di \quad = \quad -\frac{a\left(1-\gamma\right)}{1-a\left(1-\gamma\right)}\zeta,$$

which is identical to (21). However, now we need two optimality restrictions to determine a and b. These are

$$E[p_i(\epsilon_i - E[\epsilon_i|p_i,\omega_{i,j}])] = 0 \;\Rightarrow\; \gamma\sigma_\epsilon - a\left(\gamma^2\sigma_\epsilon + \frac{(1-\gamma)^2}{\left(1-a\left(1-\gamma\right)\right)^2}\sigma_\zeta\right) - b\gamma\sigma_\epsilon = 0,$$
$$E[\omega_{i,j}(\epsilon_i - E[\epsilon_i|p_i,\omega_{i,j}])] = 0 \;\Rightarrow\; \sigma_\epsilon - a\gamma\sigma_\epsilon - b\left(\sigma_\epsilon+\sigma_\eta\right) = 0,$$

which identify the equilibrium a and b such that each piece of information is orthogonal with the forecast error. Solving the system for a, we get a fix point equation written as

$$a = \frac{\gamma}{\gamma^2 + \frac{(1-\gamma)^2}{(1-a(1-\gamma))^2}\frac{\sigma_\epsilon+\sigma_\eta}{\sigma_\eta}\frac{\sigma_\zeta}{\sigma_\epsilon}}. \tag{38}$$

For $\sigma_\eta \to \infty$, the right-hand side of the relation above matches (25). In particular, it follows that a lower σ_η in (38) is equivalent to considering a larger σ_ζ in (25).

5.2 Model with local money shocks

In this section we describe a monetary economy in which local shocks are captured by variations in the local supply of money. This version of the model is distinct from our baseline model (and subsequent extensions) because it assumes that agents belongs to local families whose resources differs because of local wealth shocks.

There is a representative family on each island i with utility function

$$\log C_i - \phi N_i + \varphi \log\left(M_i/P\right) \tag{39}$$

where φ is a constant parameter and M_i is local holding of money. The local family is subject to the constraint

$$M_i + P_iC_i = W_iN_i + QZ_{(i)} + \int \Pi_i di + e^{\mu_i}M_-, \tag{40}$$

where $e^{\mu_i}M_-$ represents a stochastic endowment of money available in island i. All other variables in the model have the same meaning as before.

Given the Cobb-Douglas properties of the production function and money market clearing, the local budget constraint does not affect the worker-producers' or consumers' optimality conditions. However, to allow this simplification we need to specify how revenues from selling capital are distributed across islands. We assume that the local supply of capital is provided by a local family member who pays a lump sum, δ, for the option to supply whatever quantity is locally demanded at the market price. The ownership of Z is still equally shared across islands. As a result, the revenues of each local family are the revenues from local intermediation, $Z_{(i)}Q - \delta$, plus the revenues from common ownership, δ. Clearly, δ does not matter for aggregation; what is crucial is that revenues from the local trade of capital contribute to the family budget, allowing the choices of consumers to clear. This assumption greatly simplifies the solution of the model.[11]

The maximization problems of the representative family members become

$$\begin{aligned}
\text{money holder} &: \max_{M_i} \{\varphi \log(M_i) - \Lambda_i M_i\} \\
\text{worker-producer} &: \max_{N_i, Z_{(i)}} \left\{ P_i N_i^{\gamma} \left(e^{\zeta} Z_{(i)}\right)^{1-\gamma} - W_i N_i - Q Z_{(i)} \right\} \\
&\quad \max_{N_i} \{E[\Lambda_i | W_i] W_i N_i - \phi N_i\} \\
\text{consumer} &: \max_{C_i} \{\log C_i - E[\Lambda_i | P_i] P_i C_i\}
\end{aligned}$$

where Λ_i denotes the Lagrange multiplier associated with the budget constraint (40). The money holder's first-order condition is

$$\frac{\varphi}{e^{\mu_i} M} = \Lambda_i, \tag{41}$$

where we have already substituted in the market-clearing condition $M_i = e^{\mu_i} M$. Therefore, we can rewrite the system of first-order conditions in log terms exactly as (7)-(10). Our analysis directly follows.

[11] Other modeling strategies could achieve the same outcome. In a dynamic model, the most natural candidate for local budget clearing would be savings. We opted for a static environment in order to provide the most transparent presentation of our mechanism. We view this abstraction as relatively benign, given that our focus is on generating the shorter-lived fluctuations associated with demand shocks.

5.3 Convexity in labor disutility

Our basic framework easily extends to the case of convex disutility in labor. Let the household utility function be

$$\int e^{\mu_i} \left(\log C_i - \phi N_i^{1+\alpha}\right) di \tag{42}$$

where $\alpha > 0$ denotes the inverse of the Frisch elasticity of labor supply. In this case the local wage will not be a direct measure of the island-specific shock, but rather will depend on this shock and the aggregate quantity of labor supplied to the market. In Appendix A.1 we report the detailed derivation. Below we describe how this change affects the characterization of the equilibrium.

Characterization of the equilibrium, extended case. *An equilibrium is characterized by a profile of consumers' expectation* $\{E[\mu_i|p_i]\}_{i=0}^1$ *so that, given (12), in each island* $i \in (0,1)$ *we have*

$$p_i = \frac{\gamma}{1+\alpha}\mu_i + \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i] + (1-\gamma)(q-\zeta), \tag{43}$$

$$c_i = \frac{1+\alpha(1-\gamma)}{1+\alpha}E[\mu_i|p_i] - \frac{\gamma}{1+\alpha}\mu_i - (1-\gamma)(q-\zeta), \tag{44}$$

$$w_i = \mu_i + \alpha n_i \tag{45}$$

$$n_i = \frac{1}{1+\alpha}(E[\mu_i|p_i] - \mu_i), \tag{46}$$

$$z_{(i)} = E[\mu_i|p_i] - q. \tag{47}$$

A rational expectations equilibrium is one for which consumers' expectations, $E[\mu_i|p_i]$*, are rational.*

Proof. Derivations are provided in Appendix A.1. ∎

In this extension, the local price is affected by the individual expectation of the representative local consumer, as the equilibrium quantities of labor depend on consumers' demand. One can easily show that our analysis of the baseline economy also applies in this case, once the price signal is appropriately transformed. To arrive at a signal structure that is isomorphic to the baseline economy, subtract the individual expectation from (43) and rescale to obtain

$$\hat{p}_i = \frac{1+\alpha}{1+\alpha(1-\gamma)}\left(p_i - \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i]\right) = \hat{\gamma}\mu_i + (1-\hat{\gamma})(q-\zeta),$$

where $\hat{\gamma} = \gamma/(1+\alpha(1-\gamma))$. The analysis of Section 3 follows after substituting the original price signal, p_i, with the corresponding $\hat{p}_i$.

The extension delivers two important additional insights. First, multiple equilibria exist whenever $\hat{\gamma} < 1/2$, which could well obtain even with $\gamma > 1/2$ for a sufficiently high α. This is desirable since typical estimates of the labor share imply $\gamma > 1/2$, and might have otherwise precluded the strongest informational multipliers from appearing in a realistic calibration of the model. Second, it follows from equation (45) that demand-driven fluctuations now also feature positive comovement of wages with the average consumption price, the price of capital, total output, and total employment. The economy thus generates a robust and realistic pattern of comovement across many variables.

5.4 Equilibria selection under adaptive learning

Here, we examine the issue of out-of-equilibrium convergence, that is, whether or not an equilibrium is a rest point of a process of revision of beliefs in a repeated version of the static economy. We suppose that agents behave like econometricians. At time t they set a weight $a_{i,t}$ that is estimated from the sample distribution of observables collected from past repetitions of the economy.

Agents learn about the optimal weight according to an optimal adaptive learning scheme:

$$a_{i,t} = a_{i,t-1} + \gamma_t \; S_{i,t-1}^{-1} \; p_{i,t} \left(\mu_{i,t} - a_{i,t-1}p_{i,t}\right) \tag{48}$$

$$S_{i,t} = S_{i,t-1} + \gamma_{t+1}\left(p_{i,t}^2 - S_{i,t-1}\right), \tag{49}$$

where γ_t is a decreasing gain with $\sum \gamma_t = \infty$ and $\sum \gamma_t^2 = 0$, and matrix $S_{i,t}$ is the estimated variance of the signal. A rational expectations equilibrium $\hat{a}$ is a locally learnable equilibrium if and only if there exists a neighborhood $\mathcal{F}(\hat{a})$ of $\hat{a}$ such that, given an initial estimate $a_{i,0} \in \mathcal{F}(\hat{a})$, then $\lim_{t\to\infty} a_{i,t} \overset{a.s}{=} \hat{a}$; it is a globally learnable equilibrium if convergence happens for any $a_{i,0} \in \mathbb{R}$.

The asymptotic behavior of statistical learning algorithms can be analyzed by stochastic approximation techniques (for details, refer to Marcet and Sargent, 1989a,b and Evans and Honkapohja, 2001). The learnability analysis is postponed to appendix A.2. We show that the relevant condition for stability is $a_i'(a) < 1$, which can easily checked by inspection of Figure 1.

It turns out that the unique equilibrium is globally learnable, that is, no matter the initial estimate, revisions will lead agents to coordinate on the equilibrium. In case of multiplicity, the high and low equilibrium are locally learnable, whereas the middle equilibrium is not. Hence the middle equilibrium works as a frontier between the basins of attraction of the two equilibria.

5.5 A theory of Giffen goods?

One possible objection to the realism of our mechanism is the implication that the consumption of island-specific good C_i is rising in its price, i.e., that local consumption goods appear to be Giffen goods. Such behavior at the good level is not an essential aspect of our story. The most natural way to avoid this complication is to presume that, *within* islands, quantity-choosing firms produce a continuum of goods indexed by (i, j), which are then aggregated at the island-level good by a standard Dixit-Stiglitz aggregator, $C_i = \left(\int C_{i,j}^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}} dj$ with $\theta > 1$.

Suppose now that each (i, j) producer is hit with an idiosyncratic, mean-zero productivity shock, $\upsilon_{i,j}$. In this case, the price of good $c_{i,j}$ in logs turns out to be

$$p_{i,j} = \upsilon_{i,j} + \gamma\mu_i + (1-\gamma)(q-\zeta).$$

Demand for good $c_{i,j}$ is governed by the standard formula

$$c_{i,j} = -\theta(p_{i,j} - p_i) + c_i,$$

which reflects a substitution effect governed by the standard elasticity parameter *at the good level:* An econometrician studying good-level prices would find no evidence that the typical good is Giffen. Nevertheless, the total price level on island i,

$$p_i = \int p_{i,j} dj = \gamma\mu_i + (1-\gamma)(q-\zeta),$$

is both (i) identical to its value in the baseline economy, and (ii) reflects the optimal (even) weighting of the signals $p_{i,j}$ that consumers use in equilibrium to infer their local shock: Subsequent analysis of the island-level and aggregate economy is not affected.

6 Conclusion

Endogenous structures of asymmetric information can deliver strong multipliers on common disturbances, and thus offer a potential foundation for expectations-driven economic fluctuations. Here we have demonstrated that this mechanism can simultaneously capture several of the empirical regularities regarding business cycle fluctuations. Because of the amplification power of this mechanism, sentiment equilibria may, paradoxically, originate from economic fundamentals themselves and need not originate with shocks disconnected from the physical environment. Instead, expectations-driven fluctuations can be initiated by small changes in fundamentals that under full information would trigger far smaller, and qualitatively different, reactions.

The mechanism behind this result is a strong feedback loop that arises when agents observe, and draw inference from, endogenous variables. We microfounded such endogenous signals as competitive prices. The essential features for our mechanism are (i) shocks to local demand conditions and (ii) agents that learn from prices that reflect a combination of aggregate and local conditions. Our approach provides a natural foundation for the correlated signal structures that can produce sentiment-driven fluctuations.

A Appendix

A.1 Derivations of the model

In this section we derive equilibrium in a version of the model extended to include convexity in the disutility of labor and money in the utility function. The baseline model in the text is obtained in the cashless limit with linear disutility of labor. The extended model demonstrates our claim in the text that using the Lagrange multiplier as a numeraire is equivalent to a more standard monetary numeraire.

The utility function of the representative family is

$$\int e^{\mu_i} \left(\log C_i - \phi N_i^{1+\alpha}\right) di + \varphi \log\left(M/P\right) \tag{50}$$

and the budget constraint is

$$M + \int P_i C_i di = \int W_i N_i di + QZ + \int \Pi_i di + M_-, \tag{51}$$

where $\alpha > 0$ is the inverse of the Frisch labor elasticity, φ is a constant parameter, M is the money holding of the family, and M_- is an exogenous initial endowment of the numeraire good “money.” Money choice is delegated to a family member, the “money holder”.

The maximization problems of the representative family members become

$$\begin{aligned} \text{money holder} \quad &: \quad \max_{M} \left\{\varphi \log(M) - \Lambda M\right\} \\ \text{producer-worker} \quad &: \quad \max_{N_i, Z_{(i)}} \left\{P_i N_i^{\gamma} \left(e^{\zeta} Z_{(i)}\right)^{1-\gamma} - W_i N_i - Q Z_{(i)}\right\} \\ & \quad\;\; \max_{N_i} \left\{E[\Lambda | W_i] W_i N_i - e^{\mu_i} \phi N_i^{1+\alpha}\right\} \\ \text{consumer} \quad &: \quad \max_{C_i} \left\{E[e^{\mu_i} | P_i] \log C_i - E[\Lambda | P_i] P_i C_i\right\} \end{aligned}$$

where Λ denotes the Lagrange multiplier associated with the budget constraint (40).

The money holder’s first-order condition

$$\frac{\varphi}{M} = \Lambda, \tag{52}$$

fixes the Lagrangian to a constant as market-clearing implies that $M = M_-$. The static structure of the economy means no further assumptions are needed to prevent rational bubbles from forming in the market for money. When the exogenous money supply is fixed *ex ante*, Λ is constant so that normalizing it is equivalent to assuming a monetary numeraire.

The other log-linear first-order conditions of the economy are given by:

$$\begin{aligned}
w_i &= p_i + (\gamma - 1)n_i + (1-\gamma)z_{(i)} + (1-\gamma)\zeta \\
q &= p_i + \gamma n_i - \gamma z_{(i)} + (1-\gamma)\zeta \\
c_i &= \gamma n_i + (1-\gamma)z_{(i)} + (1-\gamma)\zeta \\
w_i &= \mu_i + \alpha n_i \\
c_i &= E[\mu_i|p_i] - p_i
\end{aligned}$$

plus the market-clearing condition for capital $\int z_{(i)}di = 0$. Fixing $\alpha = 0$, these first order conditions are identical to the baseline version of the model. Moreover, notice that in the cashless limit of this economy, i.e. in the limit $\varphi \to 0$ with the utility weight φ and money supply M_{-} in constant proportion, (50) and (40) exactly match their analogues in the baseline model.

Aggregate variables. Averaging the two sides of the labor supply condition, we have $w = \alpha n$. Thus, we have

$$\begin{aligned}
\alpha n &= p + (\gamma - 1)n + (1-\gamma)\zeta \\
q &= p + \gamma n + (1-\gamma)\zeta \\
c &= \gamma n + (1-\gamma)\zeta \\
c &= \int E[\mu_i|p_i] - p.
\end{aligned}$$

This is a linear system in four unknowns p, q, n, c, which can be expressed as functions of two states $\int E[\mu_i|p_i]di, \zeta$. Writing in matrix notation, we have

$$\begin{bmatrix} p \\ q \\ n \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1-\gamma+\alpha & 0 \\ 1 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma^{-1} \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ n \\ c \end{bmatrix} + \begin{bmatrix} 0 & \gamma - 1 \\ 0 & 1-\gamma \\ 0 & (\gamma-1)\gamma^{-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \int E[\mu_i|p_i]di \\ \zeta \end{bmatrix},$$

whose solution is

$$\begin{aligned}
p &= \frac{1+\alpha-\gamma}{1+\alpha} \int E[\mu_i|p_i]di - (1-\gamma)\zeta \\
q &= \int E[\mu_i|p_i]di \\
n &= \frac{1}{1+\alpha} \int E[\mu_i|p_i]di \\
c &= \frac{\gamma}{1+\alpha} \int E[\mu_i|p_i]di + (1-\gamma)\zeta.
\end{aligned}$$

Island-specific variables. The relevant system of equations is

$$\begin{aligned}
E[\mu_i|p_i]-c_i &= p_i\\
c_i &= \gamma n_i+(1-\gamma)z_{(i)}+(1-\gamma)\zeta\\
w_i &= p_i+(\gamma-1)n_i+(1-\gamma)z_{(i)}+(1-\gamma)\zeta\\
q &= p_i+\gamma n_i-\gamma z_{(i)}+(1-\gamma)\zeta,
\end{aligned}$$

which constitutes a linear system in four unknowns $p_i, c_i, n_i, z_{(i)}$ that can be expressed as functions of four states $\mu_i, \zeta, q, E[\mu_i|p_i]$. This system can be written as

$$\begin{bmatrix} p_i\\ c_i\\ n_i\\ z_{(i)}\end{bmatrix}=\begin{bmatrix} 0 & -1 & 0 & 0\\ 0 & 0 & \gamma & 1-\gamma\\ \frac{1}{1+\alpha-\gamma} & 0 & 0 & \frac{1-\gamma}{1+\alpha-\gamma}\\ \gamma^{-1} & 0 & 1 & 0\end{bmatrix}\begin{bmatrix} p_i\\ c_i\\ n_i\\ z_{(i)}\end{bmatrix}+\begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 1-\gamma & 0 & 0\\ \frac{1}{1+\alpha-\gamma} & -\frac{1-\gamma}{1+\alpha-\gamma} & 0 & 0\\ 0 & (1-\gamma)\gamma^{-1} & -\gamma^{-1} & 0\end{bmatrix}\begin{bmatrix} \mu_i\\ \zeta\\ q\\ E[\mu_i|p_i]\end{bmatrix}$$

where we already used $w_i=\mu_i+\alpha\mu_i$. The solution of the system is

$$\begin{aligned}
c_i &= -\frac{\gamma}{1+\alpha}\mu_i+\frac{1+\alpha(1-\gamma)}{1+\alpha}E[\mu_i|p_i]-(1-\gamma)(q-\zeta)\\
p_i &= \frac{\gamma}{1+\alpha}\mu_i+\frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i]+(1-\gamma)(q-\zeta)\\
n_i &= \frac{1}{1+\alpha}\left(E[\mu_i|p_i]-\mu_i\right)\\
z_{(i)} &= -q+E[\mu_i|p_i],
\end{aligned}$$

which is consistent with the expression for their relative aggregate variables. In the case $\alpha\neq 0$, notice that the price signal can equivalently be written as

$$\tilde{p}_i=\frac{1+\alpha}{1+\alpha(1-\gamma)}\left(p_i-\frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i]\right)=\tilde{\gamma}\mu_i+(1-\tilde{\gamma})(q-\zeta),$$

where $\tilde{\gamma}=\gamma/(1+\alpha(1-\gamma))$. Notice that limit sentiment equilibria now exist with $\tilde{\gamma}<1/2$, which could well obtain even with $\gamma>1/2$ for a sufficiently high α.

A.2 Proofs of Propositions

Proof of Proposition 1. To prove uniqueness for $\gamma\geq 1/2$, observe that the function $a_i(a)$ is continuous, bounded above by γ^{-1}, and monotonically decreasing in the range $(-\infty,(1-\gamma)^{-1})$. From $\gamma\geq 1/2$, we have $(1-\gamma)^{-1}>\gamma^{-1}$. Thus $a_i(a)$ intersects the 45-degree line a single time.

To prove the existence of a_-, notice that $\lim_{a\to-\infty}a_i=\gamma^{-1}$ and $a_i((1-\gamma)^{-1})=0$. By continuity, an equilibrium $a_-\in(0,(1-\gamma)^{-1})$ must always exist. Moreover a_- must be monotonically decreasing in σ^2 as a_i is monotonically decreasing in σ^2.

We now assess the conditions under which additional equilibria may also exist. Because $\lim_{a\to\infty}a_i=\gamma^{-1}$, the existence of a second equilibria (crossing the 45-degree line in Figure 1) implies the existence of a third. Thus, we must determine whether the difference $a_i(a)-a$

is positive anywhere in the range $a > (1-\gamma)^{-1}$. Such a difference is positive if and only if

$$\Phi(\sigma) \equiv \gamma (1 - a(1-\gamma))^2 (1-\gamma a) - a(1-\gamma)^2 \sigma^2 > 0, \tag{53}$$

which requires $a < \gamma^{-1}$ as a necessary condition. Therefore, if two other equilibria exist they must lie in $((1-\gamma)^{-1}, \gamma^{-1})$. Fixing $a \in ((1-\gamma)^{-1}, \gamma^{-1})$, $\lim_{\sigma \to 0} \Phi(\sigma)$ is positive, implying that there always exists a threshold $\bar{\sigma}$ such that two equilibria $a_+, a_\circ \in ((1-\gamma)^{-1}, \gamma^{-1})$ exist with $a_+ \geq a_\circ$ for $\sigma^2 \in (0, \bar{\sigma}^2)$. ∎

Proof of Proposition 2. Notice that $\partial\Gamma/\partial a > 0$ if and only if $\gamma < \min\{(1-\gamma)^{-1}, \gamma^{-1}\}$. The left-hand side of the fixed-point expression in (25) is downward-sloping in a and falling in σ, implying that the fixed-point intersection a_u and a_- must increase as σ falls. Similarly, $a_\circ$ falls and a_+ grows as σ falls, which implies amplifying feedback for the former and dampening feedback for the latter. ∎

Proof of Proposition 3. To prove the limiting statement for $\gamma \geq 1/2$, consider any point $a_\delta = \frac{1-\delta}{1-\gamma}$ such that $\delta > 0$. We then have

$$a_i(a_\delta) = \frac{\gamma\delta^2}{\gamma^2\delta^2 + \sigma^2(1-\gamma)^2.} \tag{54}$$

Since $\lim_{\sigma^2 \to 0} a_i(a_\delta) = \frac{1}{\gamma}$ for any δ, the unique equilibrium must converge to the same point. That the variance of this equilibrium approaches zero follows from equation (21).

To prove the limiting statement for $\gamma < 1/2$, recall the monotonicity of $a_i(a)$ on the range $(0, (1-\gamma)^{-1}$. Following the logic of Proposition 1, for any point a_δ in that range, $\lim_{\sigma^2 \to 0} a_i(a_\delta) = \gamma^{-1}$, while $a_i((1-\gamma)^{-1}) = 0$. Thus, the intersection defining a_- must approach $(1-\gamma)^{-1}$. An analogous argument for the point just to the right of $(1-\gamma)^{-1}$ establishes that a_- converges to the same value. Finally, the bounded monotonic behavior of $a_i(a)$ establishes that $\lim_{\sigma^2 \to 0} a_+ = \gamma^{-1}$ for the high equilibrium.

That the output variance of the high equilibrium in the limit $\sigma \to 0$ is zero follows from equation (22). The limiting variance of the two other limit equilibria can be established by noticing that (25) implies

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = \frac{\gamma(1-a\gamma)}{(1-\gamma)} \tag{55}$$

which, substituted into (22), gives (27) for $a \to (1-\gamma)^{-1}$. ∎

A.3 Extensions

Correlation in island-specific shocks

We now consider a version of the model in which local shocks are correlated—that is $\mu_i = \mu + \epsilon_i$ where $\mu \sim N(0, \sigma_\mu^2)$—and there are no productivity shocks. Notice that previously, productivity shocks acted as noise in the signal, since consumers were only interested in the forecast of μ_i. Now, the aggregate term μ represents a common objective in the signal extraction problem of consumers.

Following the derivation of (13), the price signal is expressed as

$$p_i = \gamma(\mu + \epsilon_i) + (1-\gamma)\int E[\mu + \epsilon_i | p_i] di, \tag{56}$$

which no longer embeds a productivity shock. Nonetheless, correlated fundamentals generate confusion between the idiosyncratic and common components of the signal. As before, the individual expectation of a consumer of type i is formed according to the linear rule $E[\mu + \epsilon_i|p_i] = a_i p_i$. Hence, the signal embeds the average expectation, which again causes the precision of the signal to depend on the average weight a. Following the analysis of the earlier section, the realization of the price signal can be rewritten as

$$p_i = \gamma\epsilon_i + \frac{\gamma}{1-a(1-\gamma)}\mu, \tag{57}$$

where a represents the average weight placed on the signal by other consumers. The variance of the average expectation is given by

$$\sigma_q^2(a) = \left(\frac{\gamma a}{1-a(1-\gamma)}\right)^2 \sigma^2, \tag{58}$$

which is slightly different from (22). The consumer's best response function is now given by

$$a_i(a) = \frac{1}{\gamma}\left(\frac{(1-a(1-\gamma))^2 + (1-a(1-\gamma))\,\sigma^2}{(1-a(1-\gamma))^2 + \sigma^2}\right). \tag{59}$$

While the best-response function in equation (59) is slightly different than that of equation (25) for the case with productivity shocks, we can prove that the characterization of the limit equilibria is identical.

Proposition 5. *In the limit $\sigma_\mu^2 \to 0$, the equilibria of the economy converge to the same points as the baseline economy:*

$$\lim_{\sigma_\mu^2 \to 0} a_e^\mu = \lim_{\sigma^2 \to 0} a_e \qquad \lim_{\sigma_\mu^2 \to 0} \sigma^2(a_e^\mu) = \lim_{\sigma^2 \to 0} \sigma^2(a_e) \qquad \text{for } e \in \{u, -, \circ, +\} \tag{60}$$

Proof. We can prove that a sentiment-free equilibrium with no aggregate variance exists for $a = \gamma^{-1}$ by simple substitution in (59). The limiting variance of the other limit equilibrium at the singularity $a \to (1-\gamma)^{-1}$ can be established by noticing that (59) implies that

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = \frac{1-a\gamma}{a\gamma} + \frac{1-a(1-\gamma)}{a\gamma}\frac{\sigma^2}{(1-a(1-\gamma))^2},$$

which gives

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = -\frac{1-a\gamma}{1-a}.$$

Substituted into (58), this gives (27) for $a \to (1-\gamma)^{-1}$. ∎

More generally, it is possible to show that Propositions 1 through 3 follow identically, and their proofs proceed in parallel with only the obvious algebraic substitutions.

Stability Analysis

To check local learnability of the rational expectations equilibrium, suppose we are already close to the resting point of the system. That is, consider the case $\int \lim_{t\to\infty} a_{i,t}\, di = \hat{a}$, where $\hat{a}$ is one of the equilibrium points $\{a_-, a_\circ, a_+\}$, and so

$$\lim_{t\to\infty} S_{i,t} = \sigma_s^2\left(\hat{a}\right) = \gamma^2\sigma_\mu^2 + \frac{\left(1-\gamma\right)^2}{\left(1-\hat{a}\left(1-\gamma\right)\right)^2}\sigma_\zeta^2. \tag{61}$$

According to stochastic approximation theory, we can write the associated ODE governing the stability around the equilibria as

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= \int \lim_{t\to\infty} \mathrm{E}\left[S_{i,t-1}^{-1} p_{i,t}\left(\mu_{i,t} - a_{i,t-1}p_{i,t}\right)\right] di \\ &= \sigma_s^2\left(\hat{a}\right)^{-1} \int \mathrm{E}\left[p_{i,t}\left(\mu_{i,t} - a_{i,t-1}p_{i,t}\right)\right] di \\ &= \sigma_s^2\left(\hat{a}\right)^{-1}\left(\gamma\sigma_\mu^2 - a_{i,t-1}\left(\gamma^2\sigma_\mu^2 + \frac{\left(1-\gamma\right)^2}{\left(1-a_{t-1}\left(1-\gamma\right)\right)^2}\sigma_\zeta^2\right)\right) \\ &= a_i\left(a\right) - a. \end{aligned} \tag{62}$$

For asymptotic local stability to hold, the Jacobian of the differential equation in (62) must be less than zero at the conjectured equilibrium. The derivative of $a_i(a)$ with respect to a is given by:

$$a_i'(a) = -\frac{2\gamma\left(1-\gamma\right)^3\left(1-\left(1-\gamma\right)a\right)\sigma^2}{\left(\left(1-\gamma\right)^2\sigma^2 + \left(1-\left(1-\gamma\right)a\right)^2\gamma^2\right)^2}, \tag{63}$$

which is positive whenever $a > (1-\gamma)^{-1}$. Then, necessarily, $a_i'(a_\circ) > 1$, $a_i'(a_+) \in (0,1)$, $a_i'(a_-) < 0$ and $a_i'(a_u) < 0$. This proves that the low and unique equilibrium are respectively locally and globally learnable.

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